SETS RELATIONS FUNCTIONS

an introduction

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Preface

The advent of the space age has created national concern for a more effective mathematics program in our schools. Several professionally recognized committees such as the Commission on Mathematics of the College Entrance Examination Board (C.E.E.B.), the School Mathematics Study Group (S.M.S.G.), the Committee on the Undergraduate Program (C.U.P.), and many others have become busily engaged in suggesting reforms to the present mathematics curriculum.

A primary aim of the recommendations proposed is to make an integrated course in calculus and analytics a standard beginning course for a college freshman program. To institute the necessary changes will require a rearrangement in the academic experiences of the student and, in many instances, a retraining of present elementary and secondary teaching personnel. Concepts and ideas previously reserved for more advanced courses in mathematics will have to be reshuffled and placed in courses at lower levels.

The concept of a set becomes important to the elementary teacher if the principles of arithmetic starting from kindergarten and upward are to be taught more effectively. The secondary school teacher must become familiar with sets as a communication medium for presenting algebra and geometry in a better fashion. He must be familiar with ideas such as the use of inequalities—absolute value—the number system viewed as a structure—postulational proof—the ordered pair and its implications for explaining more adequately the definition of a function versus a relation—and many other such notions.

The section titles of the chapters in this book indicate the type of material stressed. This text is not an "end-all." The primary objective of the authors is to present those ideas and symbolisms from set theory that will aid the reader to develop a keener insight into what has already been experienced in his mathematical background. The material is developed so as to provide him with a broader base of understanding in order to reach higher levels of abstraction. The authors believe that the utilization of the language and concepts of set theory should not be an end in itself, but its use as presented in this text should prove fruitful in terms of understanding, appreciation, and enthusiasm to other areas in mathematics.

A distinctive feature of this text is that after set language and set symbolisms are introduced in Chapter 1, this medium is not abandoned as a communication means for presenting other mathematical concepts in the remaining portion of the book. The usual criticism leveled at other corresponding texts is either that too much is included on set theory so that the reader is lost in its complexities or that, after an initial presentation, no further application of set theory is made to actual problem material in algebra, geometry, and other mathematical experiences of the reader.

The text contains approximately 200 worked-out examples and 150 graphical representations. Each new idea is presented with several illustrations, and if it is a basic concept it is reemphasized before being integrated into a newer idea. As a consequence, the authors may be accused of repetition, but from a pedagogical point of view they stand ready to accept this criticism. The exercise material involves about 1200 problems for which an answer section to selected problems is included. Scattered throughout the text are various supplementary exercises referred to as projects. These project exercises were inspired by the authors' experiences with groups of elementary and secondary teachers who were participants of National Science Foundation institute programs held at the University of Akron during the years 1960–1961 and 1961–1962. This type of project problem proved both interesting and challenging.

The expository material as presented may be used either for its own sake as a text in a senior high school class or as a supplementary reference book to other current standard high school or first-year college texts in mathematics. It is especially serviceable as text material for teachers of both the elementary and secondary levels where local school systems are conducting in-service programs or National Science Foundation institute programs.

The authors wish to express their sincere appreciation to Sister Mary Ferrer, St. Xavier College, Chicago, Illinois, to whom they are most grateful for the reading of the first and second versions of the manuscript and for the constructive suggestions which were incorporated into the final form. Thanks are also due to Miss Sally Haake, a mathematics major at the University of Akron, who typed and critically read the manuscript and prepared all the figures contained therein.

The authors take full responsibility for any shortcomings that may be found in this material—they welcome constructive criticisms from its users.

Samuel Selby
Leonard Sweet

Contents

List of Symbols

$\in$	is an element of, belongs to
$\notin$	is not an element of, does not belong to
N	set of natural numbers
P	set of primes
I	set of integers
F	set of rational numbers
R_e	set of real numbers
C	set of complex numbers
$\mid$ or $:$	such that, for which
$\leftrightarrow$	is equivalent to
$=$	is equal to
C_x	defining condition involving x
$\neq$	is not equal to
$\subseteq$	is included in
$\nsubseteq$	is not included in
$\subset$	is a proper subset of
$\not\subset$	is not a proper subset of
$\supset$	is a superset of
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\rightarrow$	implies
U	universe
$\emptyset$	null set or empty set
$n(A)$	number of elements in set A
$\cup$	union
$\cap$	intersection
A'	complement of set A
$\wedge$	and
$\vee$	or (inclusive)
$\underline{\vee}$	or (exclusive)

$\|a\|$	absolute value of a
$[a,b]$	$\{x \in R_e \mid a \leq x \leq b\}$
$[a,b[$	$\{x \in R_e \mid a \leq x < b\}$
$]a,b]$	$\{x \in R_e \mid a < x \leq b\}$
$]a,b[$	$\{x \in R_e \mid a < x < b\}$
(x,y)	ordered pair $\leftrightarrow$ coordinates of a point in the plane
$A \times B$	cartesian product of set A with set B
P_{xy}	defining condition involving x and y
R	relation
R'	complement of the relation R
R^{-1}	inverse of the relation R
f	function
f^{-1}	inverse of the function f
$f(x)$	value of the function f at x
D^*	domain
R^*	range
$f + g$	sum function
$f - g$	difference function
fg	product function
$\frac{f}{g}$	quotient function
$f \circ g$	composite function
$*$ in $a * b$	operation
$\equiv$	is congruent modulo
$\cong$	is congruent to

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1

The Vocabulary and Symbolism of Sets

1.1 INTRODUCTION

Certain ideas in mathematics, because of their scope and simplicity, constitute reservoirs of untold richness. The theory of sets, which is one such idea, was developed by Georg Cantor (1845–1918). Few concepts in the past hundred years have had as great an impact on mathematics as has the notion of a set. Set theory has contributed a foundation which clarifies and unifies the mathematics already developed. It provides a language and a symbolism which make it possible to synthesize the old and the new, to examine familiar concepts, and to view new and exciting milestones along the mathematical highway. To reach the first milestone on the highway, a familiarity must be established with the vocabulary and symbolism of set theory—the objective of Chapter 1.

1.2 CONCEPT OF A SET

A set is any well-defined collection of objects. Other words, such as collection, class, and aggregate, are used synonymously with the term set. "Well-defined" means that it is possible to determine readily whether an object is a member of a set or not. For example, the set of the five greatest living statesmen is not a well-defined set. This is so because the criteria or standards as to what determines a great living statesman are not commonly agreed upon by everyone. Given the name of a particular statesman, we should have difficulty in determining definitely whether this individual is a member of or is not a member of the set. However, with the set of 50 states of the United States no difficulty would be experienced in determining definitely whether any given object is or is not contained in this set.

The individual objects that belong to a set are called its elements. If capital letters A, B, C, . . . denote sets and small letters a, b, c, . . . represent elements, then the notation "$a \in A$" is read "a belongs to A"

or "a is an element of A." "$b \notin B$" is read "b does not belong to B." The symbol $\in$ (belongs to) or $\notin$ (does not belong to) is referred to as the "membership relation." The notation $x_1, x_2, x_3, \ldots, x_n \in A$ means that each $x_i \in A$.

Sets may also be collections of sets. For example, the set of baseball teams in the National League is a set of teams where each team is an element. Further, each player is an element of the set constituting the team on which he plays.

The following examples illustrate the "concept of set" and the "membership relation."

Example 1. If G = the set of vowels in the English alphabet, then $e \in G$, but $r \notin G$.

Example 2. If B = the set of months beginning with the letter J, then January $\in B$, but May $\notin B$.

Example 3. Let N = the set of natural numbers (counting integers, excluding zero). Hence $3 \in N$ and $11 \in N$, but $\frac{3}{4} \notin N$ and $0 \notin N$.

Example 4. Let P = the set of primes (a natural number is a prime if it has two distinct divisors, itself and 1). Hence $2 \in P$ and $5 \in P$, but $1 \notin P$ and $8 \notin P$.

Example 5. Let I = the set of integers (positive and negative integers and zero). Then $0 \in I$, $3 \in I$, and $-5 \in I$; but $\frac{2}{3} \notin I$ and $\frac{9}{4} \notin I$.

Example 6. Let F = the set of rational numbers (a number is rational if it can be expressed as the quotient p/q, of two integers p and q where $q \neq 0$). Then $\frac{3}{5} \in F$, $\frac{4}{3} \in F$, and -5 or $\dfrac{-5}{1} \in F$; but $\sqrt{3} \notin F$, $\sin 12° \notin F$, and $\log 17 \notin F$.

Table 1

	N	P	I	F
0	$\notin$	$\notin$	$\in$	$\in$
1	$\in$	$\notin$	$\in$	$\in$
-2	$\notin$	$\notin$	$\in$	$\in$
$\frac{3}{4}$	$\notin$	$\notin$	$\notin$	$\in$
$-\frac{9}{4}$	$\notin$	$\notin$	$\notin$	$\in$
-8	$\notin$	$\notin$	$\in$	$\in$
π	$\notin$	$\notin$	$\notin$	$\notin$
$3\frac{1}{3}$	$\notin$	$\notin$	$\notin$	$\in$
5	$\in$	$\in$	$\in$	$\in$
$\sin 30°$	$\notin$	$\notin$	$\notin$	$\in$
$\cos\left(\frac{\pi}{10}\right)$	$\notin$	$\notin$	$\notin$	$\notin$

Example 7. If N = the set of natural numbers, P = the set of primes, I = the set of integers, and F = the set of rational numbers, then the membership relation ($\in$ or $\notin$) for each of the numbers $0, 1, -2, \frac{3}{4}, -\frac{9}{4}, -8, \pi, 3\frac{1}{3}, 5, \sin 30°$, and $\cos(\pi/10)$ is indicated in Table 1.

Example 8. Let T = the set of integers satisfying the equation $2x - 7 = 5$. Hence $6 \in T$ (note that 6 is the only element of T).

Exercise 1

1. A set must be a well-defined collection of objects. Which of the following objects form sets according to this definition?
a. The set of the three greatest musical compositions
b. The set of all months of the year beginning with the letter D
c. The set of the 10 greatest living Americans
d. The set of words appearing in this book
e. The set of the five most talented actors

2. Let Q denote the set of all the quadrilaterals of plane geometry. Using the connectives "$\in$ and $\notin$," indicate the membership relation for each of the following figures.

Example. A triangle t *Answer:* $t \notin Q$

a. A rhombus r	*b.* A square $\square$	*c.* A parallelogram p
d. A hexagon h	*e.* A rectangle $\square$	*f.* A pentagon g
g. A trapezoid z	*h.* A circle c	

3. Which of the following sets have elements that are also sets?
a. The set of football teams in the National Football League
b. The Cleveland Symphony Orchestra
c. The American Federation of Labor
d. The United Nations
e. The set of all counties in the United States

4. Give two examples of sets that are well-defined; two examples of sets that are not well-defined.

1.3 FINITE AND INFINITE SETS

The natural numbers $1, 2, 3, \ldots, n, \ldots$ represent an infinite set of elements. Given a natural number, then by adding 1 another natural number is formed. Consequently, this infinite set has no last element. This property characterizes infinite sets. Examples of infinite sets are:
a. The set of all circles in a plane
b. The set of odd primes
c. The set of integers divisible by 3
d. The set of rational numbers greater than zero
e. The set of points on a line

If a set is finite, then it has a last element. As a consequence, it is

always possible to determine the number of elements that belong to the set. Examples of finite sets are:

a. The set of months of the year (12 elements)

b. The set of letters in the English alphabet (26 elements)

c. The set of all books in the Library of Congress (nearly 12 million elements)

d. The set of all the grains of sand on the beach at Atlantic City (though this set is very large it contains a finite number of elements)

e. The set of even primes (one element)

For the purpose of generalization, a set containing no elements is defined as a finite set. A set containing no elements is called the null or empty set and is symbolized by the notation $\emptyset$ or $\{\quad\}$. Examples of null sets are:

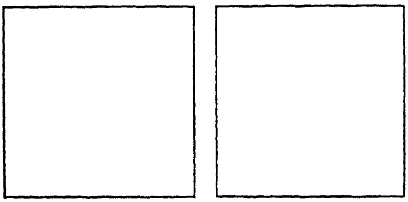


FIG. 1

a. The set of positive integers between 3 and 4

b. The set of students in your classroom whose birthplace was the planet Jupiter

c. The set of integers satisfying the equation $9x^2 - 4 = 0$

d. The set of all points that $\in$ both squares shown in Fig. 1

Exercise 2

1. Give two examples of finite sets; two examples of infinite sets.

2. Give two examples of empty sets.

3. State whether each of the following sets is finite or infinite. When the set is finite, indicate the number of elements it possesses.

a. The set of letters in the word Massachusetts
b. The set of odd positive integers
c. The set of all positive two-digit integers
d. The set of integers satisfying the equation $x^2 - 5x - 6 = 0$
e. The set of rational numbers satisfying the equation $x^2 - 5 = 0$
f. The set of quadrilaterals
g. The set of primes greater than 2 and less than 75
h. The set of students in your school who have two heads
i. The set of all lines passing through a fixed point in a plane
j. The set of intersection points of two circles in a plane

1.4 DESCRIPTION OF SETS

Two methods used frequently to describe sets are the "tabulation method" and the "defining-property method." The first, the "tabulation method," enumerates or lists the individual elements, separates them by commas, and encloses them in braces.

Example 1. $\{3,2,5\}$ is the set whose elements are the numbers 3, 2, and 5. The order of listing is unimportant; the set $\{3,2,5\}$ is identical with the set $\{5,3,2\}$. In fact, there are six ways of listing this set: $\{3,2,5\}$, $\{3,5,2\}$, $\{5,3,2\}$, $\{5,2,3\}$, $\{2,5,3\}$, and $\{2,3,5\}$. If the order is important, it must be so specified.

If a set contains only one element, it is called a unit set. $A = \{x\}$ is an example of a unit set whose only member is x. Note that $A = \{x\}$ is not the same as $A = x$.

Enumeration of an infinite set consists in listing a few elements followed by three dots. $N = \{1,2,3,4, \ldots\}$ is the notation for the infinite set of natural numbers. The same notation is used for a finite set, but the last element is always included. The set of natural numbers less than 1000 can be written as $M = \{1,2,3, \ldots ,998,999\}$.

Some sets cannot be described by an enumeration. A second method, which defines a property, is often more compact and convenient. For example, the set of rational numbers between 5 and 6 and the set of even integers between 1 and 25 are described by

$$F = \{\text{all rational numbers between 5 and 6}\}$$
$$E = \{\text{all even integers between 1 and 25}\}$$

This defining condition may take different forms. It may consist of descriptive words, symbols from mathematics, or a combination of both. The defining condition spells out specifically the requirements that an object must satisfy in order to belong to the set. The defining-property method provides, in a precise form, a test for membership of all the elements belonging to the set, which the tabulation method lacks. This is especially true with regard to those elements absent from a listing of the set.

Example 2. The set of one-digit primes could be represented by $A = \{2,3,5,7\}$ according to the tabulation method. By the defining-property method we have $A = \{x \mid x \text{ is a one-digit prime}\}$ or

$$A = \{x : x \text{ is a one-digit prime}\}$$

This is read "the set of all elements x such that x is a one-digit prime." The vertical bar "|" or ":" is read "such that" or "for which." The notation then takes the general form $\{x \mid \text{some defining condition about } x\}$ or $\{x \mid C_x\}$, where C_x represents the defining condition involving x. Frequently, the notation is modified and written

$$A = \{x \in P \mid x \text{ is a one-digit number}\}$$

Here P is the set of all primes and A is now read "the set of all those elements x of P such that x is a one-digit number."

The letter x is called a variable or placeholder. It should be noted that any other desired symbol such as y, z, a_1, a_2, $*$, $\triangle$, $\square$, or α could be used to represent the variable. This symbol holds the place for any element of the set that is being defined. The elements of the defined set are referred to as the values of the variable. When a defined set consists of just a single value, for example, π, we call the variable a constant. Thus a variable may hold the place for either a finite number of values or an infinite number of values, depending upon the defined set.

The following sentences are "defining conditions" placed on the elements:

x is an even integer.
x is an integer such that $x + 7 = 8$.
x is a natural number such that $x^2 - 3x + 2 = 0$.
x is an integer greater than 5.
x is a natural number divisible by 3.

The following examples illustrate the use of defining conditions in describing sets.

Example 3. Let R_e denote the set of real numbers, i.e., the collection of all rational and irrational numbers. Irrational numbers are numbers such as $\sqrt{2}$, $\sqrt[3]{5}$, and π which are not replaceable by quotients of two integers. Thus $\{x \in R_e \mid (x - 5)(x + 5) = 0\}$ is another way of describing the set $\{-5,5\}$, while $\{x \in R_e \mid x - \sqrt{3} = 0\}$ is another way of describing the set $\{\sqrt{3}\}$.

Example 4. Given: the condition $x^2 = -4$ and $x \in R_e$. The solution set G may be described as

$$G = \{x \mid x \in R_e \text{ and } x^2 = -4\} \qquad \text{or} \qquad G = \{x \in R_e \mid x^2 + 4 = 0\}$$

This leads to what is called the empty set, since the square of every real number is a nonnegative (zero or positive) real number. The empty set is written $\emptyset$ or $\{\quad\}$. Thus $G = \emptyset$ or $\{\quad\}$. It is important to keep in mind that 0 and $\emptyset$ do not have the same meaning from the standpoint of sets. Thus $\emptyset$ is not equal to 0 or $\{0\}$, since $\{0\}$ is a set with one element "0" while $\emptyset$ or $\{\quad\}$ is the set that contains no elements. Then "0" would not be an element in the empty set but could be used to indicate the number of elements possessed by this set.

Example 5. If $G = \{x \in R_e \mid x^2 = -4\}$, then $G = \emptyset$. However, if $H = \{x \in C \mid x^2 = -4\}$, then $H = \{2i,-2i\}$. Here C represents the set of complex numbers.

Example 6.

$$\{x \mid x = x_1 \text{ or } x = x_2 \text{ or } \cdots \text{ or } x = x_n\} = \{x_1,x_2,x_3, \ldots ,x_n\}$$

Exercise 3

1. Describe the following sets in mathematical notation. Indicate whether the set is finite or infinite. If the set is finite, indicate the number of elements that belong to the set and enumerate where possible.

a. The set of letters in the word Mississippi
b. The set of even positive integers
c. The set of real numbers satisfying $x^2 - 5x + 4 = 0$
d. The set of real numbers satisfying $x^2 + x + 1 = 0$
e. The set of two-digit negative integers
f. The set of primes greater than 6 and less than 40
g. The set of positive integers divisible by 5
h. The set of consonants in the English alphabet
i. The set of all equilateral triangles

2. Consider the set $G = \{x \mid x \text{ is a prime less than } 15\}$. The sentence "$x$ is less than 15" can be considered a "set selector," since it selects from the set of primes those elements that satisfy the given requirement. If the elements of G are enumerated, then $G = \{2,3,5,7,11,13\}$. Indicate the set selector for each of the following and describe the set by enumerating the elements.

a. $H = \{x \mid x \text{ is an integer greater than } -2 \text{ but less than } 8\}$
b. $T = \{x \mid x \text{ is a positive three-digit integer divisible by } 13\}$
c. $F = \{x \mid x \text{ is a positive even integer less than } 10\}$
d. $A = \{x \mid x \text{ is a month of the year having less than 30 days}\}$
e. $C = \{x \mid x \text{ is a positive odd integer}\}$
f. $E = \{x \mid x \text{ is a negative integer greater than } -1000\}$
g. $G = \{x \mid x \text{ is an integer and } x + 1 = x\}$
h. $K = \{x \mid x \text{ is a rational number and } x + 2x + 5 = 3x + 5\}$
i. $L = \{x \mid x \text{ is a positive integer and } 2x - 3 = 5\}$
j. $M = \{x \mid x$ is a proper fraction having one-digit numerators and denominators less than 5; all fractions reduced to lowest terms$\}$
k. $Q = \{x \mid x \in N \text{ and } 2x - 5 \text{ is less than } 6\}$
l. $S = \{x \mid x \in N \text{ and } x^2 \text{ is less than } 17\}$
m. $T = \{x \mid x \in I \text{ and } x^2 \text{ is less than } 17\}$
n. $V = \{x \mid x \in N \text{ and } 8x \text{ is greater than } x^2\}$

3. Use the defining-property method to describe each of the following sets.

Example. $A = \{1,2,3,4,5\}$ *Answer:* $\{x \in N \mid x \text{ is less than } 6\}$

a. $B = \{2,4,6,8\}$
b. $C = \{2\}$
c. $D = \{\ \}$ or $D = \emptyset$
d. $E = \{200,201, \ldots ,299\}$

4. Given: $N = \{\text{natural numbers}\}$

$$I = \{\text{all integers}\}$$

$$F = \left\{x \mid x = \frac{p}{q} \text{ where } p,q \in I \text{ and } q \neq 0\right\}$$

$$R_e = \{\text{all real numbers}\}$$

Select from the indicated set those elements, if any, which satisfy the given requirement.

Statement	N	I	F	R_e
a. $3x - 4 = 2$				
b. $3x - 1 = 4$				
c. $x^2 = 4$				
d. $x^2 = -4$				
e. $x^2 = 5$				
f. $(x - 1)(x + 1) = x^2 - 1$				
g. $x(x - 1)(x + 2) = 0$				
h. $x(x + 1) = x^2 - 1$				
i. $x^2 - 3x = 0$				

5. Indicate the weakness of using the tabulation method for describing a set specified by a defining property. Discuss this with respect to the set A described in the following two ways:

$$A = \{p \mid p \in I^+, n \in I^+, \text{and } p = n^2 + 3(n - 1)(n - 2)(2n - 1)(n - 3)\}$$

where I^+ is the set of positive integers, or

$$A = \{1,4,9, \ldots\}$$

(*Hint:* The second form could suggest the elements 16, 25, . . . as those following the listing 1, 4, 9, ...)

6. If the following refer to figures in plane geometry, determine the truth or falsity of each statement.

a. If $x \in$ {squares}, then $x \in$ {rectangles}.
b. If $x \in$ {quadrilaterals}, then $x \in$ {polygons}.
c. If $x \in$ {equilateral triangles}, then $x \in$ {equiangular triangles}.
d. If $x \in$ {isosceles triangles}, then $x \in$ {equilateral triangles}.
e. If $x \in$ {equilateral triangles}, then $x \in$ {isosceles triangles}.
f. If $x \in$ {rectangles}, then $x \in$ {parallelograms}.
g. If $x \in$ {parallelograms}, then $x \in$ {rectangles}.
h. If $x \in$ {rhombuses}, then $x \in$ {parallelograms}.
i. If $x \in$ {rectangles}, then $x \in$ {squares}.

1.5 MAPPING AND ONE-TO-ONE CORRESPONDENCE

A mapping of a set A "into" a set B is a matching procedure that assigns to each element $a \in A$ a unique element $a' \in B$. This mapping may be represented by the notation "$a \to a'$," (read, a "maps into" a') where a' is called the image of a under the designated mapping. If in this matching procedure every element of B is used as an image, then the mapping is said to be "A onto B" as well as "A into B."

Example 1. If $A = \{1,2,3\}$ and $B = \{b,c\}$, then $1 \to c$, $2 \to c$, $3 \to b$ defines a mapping of set A "onto" set B. However, $1 \to b$, $2 \to b$, $3 \to b$ defines a mapping of A "into" B, since the element $c \in B$ is not an image of any of the elements of A.

Example 2. Let $C = \{$all the names in a given telephone directory$\}$ and $D = \{$all the numbers of the pages of the directory$\}$. If each name is associated with the corresponding page number on which the name is printed, then a mapping of C onto D has been defined.

Example 3. If $A = \{1,3,5,7,9\}$ and $B = \{2,4,6,8,10\}$, then $1 \to 2$, $3 \to 4$, $5 \to 6$, $7 \to 8$, $9 \to 10$ defines a one-to-one mapping of A onto B. Here each $a' \in B$ is the image of at most one $a \in A$. The mapping may be reversed; that is, $2 \to 1$, $4 \to 3$, $6 \to 5$, $8 \to 7$, $10 \to 9$, which then defines a one-to-one mapping of B onto A. Thus each odd number in A can be said to have a distinct even number in B as its image, while in the reverse mapping each even number of B has a distinct odd number in A as its image. In this case the two sets A and B are said to be in one-to-one correspondence.

Two sets A and B are said to be in 1-1 correspondence when there exists a 1-1 mapping of A onto B. It is important to note that a 1-1 mapping of A onto B always ensures a 1-1 mapping of B onto A. If two sets A and B can be placed in 1-1 correspondence, they are said to be equivalent. The equivalence of A and B is symbolized $A \leftrightarrow B$.

Example 4. The sets $\{b,c\}$ and $\{e,f\}$ are equivalent. The mappings may be performed in the following manner:

$$\begin{array}{ccc} b \leftrightarrow e & & b \qquad e \\ & \text{or} & \\ c \leftrightarrow f & & c \qquad f \end{array}$$

Example 5. The sets $\{$Tom, Dick, Harry$\}$ and $\{$Mary, Jean$\}$ are not equivalent. Tom may be matched with Mary, and Dick with Jean, but no element of the second set remains to be paired with Harry.

Example 6. Establish a 1-1 correspondence between the set of numbers $\{1,2,3, \ldots ,26\}$ and the set of letters of the alphabet $\{a,b,c, \ldots ,z\}$. We have

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & 25 & 26 \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ a & b & c & \cdots & y & z \end{array}$$

It should be observed that there are many other possible ways to establish this 1-1 correspondence. For example,

$$\begin{array}{cccccc} 2 & 3 & 4 & \cdots & 26 & 1 \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ a & b & c & \cdots & y & z \end{array}$$

Example 7. When all the students in a classroom are seated one to a chair, a 1-1 correspondence is set up between the set of chairs and the set of students. It is being assumed that there are just as many chairs as students.

Example 8. The points on the coordinate line are in 1-1 correspondence with the set of real numbers (see Section 2.6).

Example 9. The set of points in a plane are in 1-1 correspondence with the set of ordered pairs of real numbers (see Section 3.3).

All the sets in Examples 1 through 7 are finite sets, while those of Examples 8 and 9 are infinite sets.

The primary purpose of 1-1 correspondence is to determine the equivalence of infinite sets. Two finite sets can be shown to be equivalent if they both contain the same number of elements, but the establishment of the equivalence of two infinite sets by counting would lead to failure because of the unlimited number of elements contained by these sets. However, if two infinite sets can be placed in 1-1 correspondence, then these sets are equivalent. For example, the set of natural numbers $N = \{1,2,3,4,5,6,7, \ldots, n, \ldots\}$ can be shown to be equivalent to the set of even natural numbers $E = \{2,4,6,8,10,12, \ldots, 2n, \ldots\}$ by setting up a one-to-one correspondence according to the following scheme:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots & n & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & \cdots & 2n & \cdots \end{array}$$

From this arrangement it is evident that each natural number may be matched with an even natural number. In this case, there are just as many even natural numbers as there are natural numbers, and $N \leftrightarrow E$.

Exercise 4

1. Show that the set $\{1,2,3\}$ is equivalent to the set $\{a,b,c\}$. How many ways are there of matching the elements of these two sets?

2. Give an example of two sets that are equivalent but do not contain the same elements.

3. Show that the set of natural numbers is equivalent to the set of odd natural numbers.

4. Show that the set of points on the line segment AB can be placed in 1-1 correspondence with the set of points in the base EF of the triangle EFG (Fig. 2). Show that the set of points on EF is equivalent to the set of points on GF.

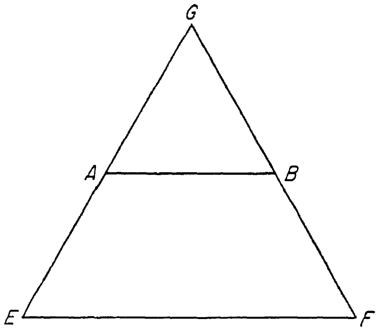


FIG. 2

5. Consider the sets $\{a,b,c\}$, {Tom, Dick, Harry}, $\{+,-,\div\}$, {apple, orange, banana}, and {flute, trumpet, violin}. These sets are all equivalent to each other, since they each have the same number of elements and can be placed in 1-1 correspondence with each other. All five sets share a common property in that the number "3" may be used to designate the number of elements in each set. Such a number is referred to as a "cardinal number" and is associated with the collection of sets, all of which are equivalent to each other. This collection of sets is called an "equivalence class of sets," and the cardinal numbers are associated with equivalence classes. The cardinal number associated with the equivalence class of sets containing the set $\{a,b,c,d\}$ and all other sets equivalent to $\{a,b,c,d\}$ is 4.

Example:

Sample element of equivalence class	*Cardinal number of the class*
$\{1,2\}$	2
$\{3,4,5\}$	3
$\{6,7,8,9\}$	4
$\{\ \} = \emptyset$	0

a. What is the cardinal number of the equivalence class of sets having the sample element $\{a\}$; the sample element $\{\square, \triangle, \bigcirc\}$; the sample element $\{1,2,3,4,5,6,7\}$.

b. List two other sets that belong to the same equivalence class as $\{\alpha,\beta,\gamma\}$; as $\{\vee,\wedge,\underline{\vee},\leftrightarrow\}$.

1.6 EQUAL SETS

When two sets A and B are equal, $A = B$, then this implies that each element of A is an element of B and that each element of B is an element of A. If either of the two sets contains a distinct element not contained in the other, then $A \neq B$ (set A does not equal set B). Note that if two sets A and B are equal, then it follows that they are equivalent, but the converse does not hold.

Example 1. If $A = \{1,2,3\}$ and the set

$$B = \{x \in R_e \mid (x - 1)(x - 2)(x - 3) = 0\}$$

then $A = B$. The order in which the elements of a set are listed has no bearing on whether $A = B$ or $A \neq B$. Thus if $C = \{1,2,4\}$ and $D = \{2,1,4\}$, then $C = D$. But if $D = \{1,2,4\}$ and $T = \{2,4\}$, then $D \neq T$ since $1 \in D$ but $1 \notin T$. It should be noted that sets specified by a defining property or a tabulation may be equal and yet outwardly look different. Hence

$$\{1,2,3\} = \{x \in R_e \mid (x - 1)(x - 2)(x - 3) = 0\}$$

Example 2. If $K = \{x \mid x \text{ is a prime less than } 7\}$ and

$$G = \{x \in R_e \mid (x - 1)(x - 2)(x - 3)(x - 5) = 0\}$$

then $K \neq G$. Here $K = \{2,3,5\}$ and $G = \{1,2,3,5\}$. $1 \notin K$, since by definition, 1 is not considered a prime.

Example 3. A set may be described through the use of different defining properties. For example, if $E = \{0,1,2,3\}$, then

$$E = \{x \mid x \in I \text{ and } x \text{ is between } -1 \text{ and } 4\}$$

or $E = \{x \mid x \text{ is the remainder when any natural number is divided by } 4\}$

or $$E = \{x \mid x \in I \text{ and } x(x - 1)(x - 2)(x - 3) = 0\}$$

Exercise 5

1. Determine whether the relation $A = B$ or $A \neq B$ holds for each of the following pairs of sets:

a. $A = \{x \mid x \in N \text{ and } x \text{ is less than } 5\}$
$B = \{x \mid x \in N \text{ and } (x + 1)^2 \text{ is less than } 28\}$
Answer: $A = B$, since $A = \{1,2,3,4\}$ and $B = \{1,2,3,4\}$.

b. $A = \{x \mid x \in N \text{ and } x \text{ is less than } 6\}$
$B = \{x \mid x \in N \text{ and } (x + 1)^2 \text{ is less than } 40\}$

c. $A = \{x \mid x \in N \text{ and } x \text{ is odd}\}$
$B = \{x \mid x \in N \text{ and } x^2 \text{ is odd}\}$

d. $A = \{x \mid x \in N \text{ and } x^2 \text{ is less than 20 and greater than } 8\}$
$B = \{x \mid x \in N \text{ and } x^2 - 7x + 12 = 0\}$

e. $A = \{x \mid x \text{ is a square with an area greater than nine square units}\}$
$B = \{x \mid x \text{ is a square with a perimeter greater than 12 linear units}\}$

f. $A = \{x \mid x \in N \text{ and } x \text{ is an even prime number}\}$
$B = \{x \mid x \in I \text{ and } x^2 - 2x = 0\}$

g. $A = \{x \mid x \text{ is a positive even integer divisible by } 5\}$
$B = \{x \mid x \text{ is an even prime number greater than } 3\}$

h. $A = \{x \mid x \text{ is a quadrilateral}\}$
$B = \{x \mid x \text{ is a polygon}\}$

1.7 SUBSET AND UNIVERSE

A set B is said to be a subset of a set A if and only if each element of B is an element of A. For example, the set of vowels is a subset of the set of all letters in the alphabet, since each vowel is a letter in the alphabet. Further, the set $B = \{1,2,3,4\}$ is a subset of the set $A = \{1,2,3,4,5,6\}$, since every element of B is also an element of A.

If B is a subset of A, then B is included in A. Every element of B is also an element of A. The notation "$\subseteq$" means inclusion. Symbolically, if "$\rightarrow$" means "implies," then $B \subseteq A \rightarrow$ if $a \in B$, then $a \in A$. If B is not a subset of A, we write $B \nsubseteq A$. For example, if $B = \{2,3\}$ and $A = \{2,3,4\}$, then $B \subseteq A$ or $\{2,3\} \subseteq \{2,3,4\}$. Further, $\{2,3\} \subseteq P$ where P is the set of primes, but $A \nsubseteq P$ since $4 \notin P$. From our definition of set inclusion, it follows that $\{2,3\} \subseteq \{2,3\}$ or $B \subseteq B$. In general, every set is a subset of itself.

In addition, the agreement is made that the null set $\emptyset$ is a subset of every set; that is, for any set A, each member of $\emptyset$ must be an element of A. Since $\emptyset$ contains no elements, the requirement for inclusion is not contradicted and we may write $\emptyset \subseteq A$.

The equality of sets may be interpreted in terms of set inclusion. Thus, if A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

A set B is a "proper subset" of a set A if and only if B is a subset of A and at least one element of A is not an element of B; i.e., $B \subseteq A$ and $B \neq A$. To indicate that B is a proper subset of A we write "$B \subset A$." When B is not a proper subset of A, we write "$B \not\subset A$." Further, by convention the entire set (in the discussion) and the null set will be referred to as "improper subsets." For example, $\{a,b\} \subset \{a,b,c,d\}$, or $\{a,b\}$ is a proper subset of $\{a,b,c,d\}$, but $\{a,b\} \not\subset \{a,b\}$. The set of even integers is a proper subset of the set of integers, and the set of primes is a proper subset of the set of natural numbers. If B is a proper subset of A, then A is called a superset of B and written $A \supset B$. Thus if $A = \{3,4,5\}$ and $B = \{3,4\}$, then $\{3,4,5\} \supset \{3,4\}$. We illustrate these important ideas with the following examples.

Example 1. Consider the following situation. You are requested to make a contribution to the United Fund. In examining your wallet you discover that you have a \$1 bill, a \$5 bill, and a \$10 bill. Using only these three elements and representing the possible contributions that you can make in terms of a tabulation method, we have $\{1\}$, $\{5\}$, $\{10\}$, $\{1,5\}$, $\{1,10\}$, $\{5,10\}$, $\{1,5,10\}$, and $\{\ \ \}$. The set $\{1,5\}$ indicates a contribution of \$6, or \$1 + \$5, while the set $\{\ \ \}$ indicates that no contribution is made. Each of these sets is a subset of the entire set $U = \{1,5,10\}$. Since $\{1,5,10\}$ is one of the possible contributions, we

agree that $\{1,5,10\}$ is also a subset of U. Similarly we agree that $\emptyset$ or $\{\ \}$ is also a subset of U or $\emptyset \subseteq U$. We have formed new sets from a complete set U, each new set containing elements that are drawn from U. The sets $\{1\}$, $\{5\}$, $\{10\}$, $\{1,5\}$, $\{5,10\}$, and $\{1,10\}$ are proper subsets of $\{1,5,10\}$, while the subsets $\{1,5,10\}$ and $\emptyset$ are improper subsets.

Example 2. Given: the following sets of plane figures:

$$\begin{aligned}
A &= \{\text{triangles}\}\\
B &= \{\text{isosceles triangles}\}\\
C &= \{\text{equilateral triangles}\}\\
D &= \{\text{equiangular triangles}\}\\
E &= \{\text{scalene triangles}\}\\
G &= \{\text{right triangles}\}\\
H &= \{\text{equilateral right triangles}\}\\
K &= \{\text{isosceles right triangles}\}
\end{aligned}$$

Using the symbols $\subset$, $=$, $\supset$, and $\not\subset$, indicate the relation between each of the following pairs of sets.

	Pair	*Relation*
a.	A,B	$A \supset B$ or $B \subset A$
b.	B,C	$C \subset B$ or $B \supset C$
c.	C,D	$C = D$
d.	A,E	$E \subset A$ or $A \supset E$
e.	B,G	$B \not\subset G$ or $G \not\subset B$
f.	E,G	$E \not\subset G$ or $G \not\subset E$
g.	H,G	$H \not\subset G$, since $H = \emptyset$
h.	K,G	$K \subset G$ or $G \supset K$
i.	B,K	$K \subset B$ or $B \supset K$

Example 3. Given: the following sets of numbers:

$$\begin{aligned}
N &= \{x \mid x \text{ is a natural number}\}\\
I &= \{x \mid x \text{ is an integer}\}\\
P &= \{x \mid x \text{ is a prime}\}\\
F &= \{x \mid x \text{ is a rational number}\}\\
R_e &= \{x \mid x \text{ is a real number}\}
\end{aligned}$$

Using the symbol $\subset$ or $\not\subset$, determine all the relations connecting these sets in pairs:

$N \subset I$	$I \subset F$	$P \subset N$	$F \subset R_e$	$R_e \not\subset N$
$N \subset F$	$I \subset R_e$	$P \subset I$	$F \not\subset N$	$R_e \not\subset I$
$N \subset R_e$	$I \not\subset P$	$P \subset F$	$F \not\subset I$	$R_e \not\subset P$
$N \not\subset P$	$I \not\subset N$	$P \subset R_e$	$F \not\subset P$	$R_e \not\subset F$

When a decision is made to form subsets, there must be some source from which the elements of these subsets are to be chosen; that is, any particular discussion of sets must be limited to some fixed set called a universal set or universe. The universe (designated as U) represents the complete set or largest set for the particular discussion, and all other sets in that same discussion will be subsets of U. It should be noted that the universal set is not the same for all discussions or problems. The choice of a universe is dependent upon the problem being considered. For example, in one case it may be the set of rational numbers; in another it may be the set of all persons in Ohio, or the set of all triangles, or the set of all points in a plane, etc.

In mathematics the defining conditions used to describe sets usually consist of an equation or inequality, such as $x + 2 = 6$ or $x > 5$, together with a universe that represents the set of possible replacements for the variable x. The universe is frequently a set of numbers such as the set of natural numbers, the set of integers, the set of rational numbers, or the set of real numbers. It is very important to specify the universe of the variable or, in other words, "the set of possible replacements for the variable." Once the universe is designated, any condition in the form of an equation or inequality separates this universe into two subsets: the set of replacements for x that satisfy the given condition and the set of replacements that do not satisfy this condition. For example, if U is the set of integers and $3x - 2 = -14$, then -4 is the only integer that yields the true statement $3(-4) - 2 = -14$. All other replacements from the set of integers result in false statements. According to the defining-property method, this set is written $\{x \in I \mid 3x - 2 = -14\}$.

The equation $3x - 2 = -14$ is called the "set selector," since it selects from the universe a set of elements that will satisfy the equation. This set of elements is called the "solution set" of the equation. The solution set of $3x - 2 = -14$ is $\{-4\}$. Sometimes the solution set is the empty set; that is, no elements from the universe satisfy the stated condition. Sometimes the solution set is identical with the universe; that is, all elements of the universe satisfy the stated condition. Thus, a solution set can contain a finite number (zero or some natural number) of elements or, at other times, an infinite number of elements. The solution set may be specified by either the defining-property method or the tabulation method. Hence, if we are given the condition $x^2 - 2x - 3 = 0$ and U = the set of integers, we may describe the solution set as either $\{x \in I \mid x^2 - 2x - 3 = 0\}$ or $\{3, -1\}$.

Example 4. The solution set of a particular condition is a subset of the universe of the variable. If the defining condition is $x < 5$ and the universe $U = \{x \mid x \in N\}$, then the solution set A may be written

$A = \{x \in N \mid x < 5\}$ or $A = \{1,2,3,4\}$. Thus A is a subset of U or $A \subseteq U$. In fact, A is a proper subset of $U = N$.

Example 5. For each of the following sets the universe of the variable, the set selector, and the solution set are:

a. $\{x \in N \mid x \text{ is an even number}\}$
Universe: Set of natural numbers or $\{x \mid x \in N\}$
Set selector: x is an even number
Solution set: $\{2,4,6,8, \ldots\}$

b. $\{x \in I \mid x^2 - x = x(x - 1)\}$
Universe: Set of integers or $\{x \mid x \in I\}$
Set selector: $x^2 - x = x(x - 1)$
Solution set: $\{\ldots, -4,-3,-2,-1,0,1,2,3, \ldots\}$ or $\{x \mid x \in I\}$, since every integer will satisfy the equation

c. $\{x \in R_e \mid x - 2 = x\}$
Universe: {real numbers} or $\{x \mid x \in R_e\}$
Set selector: $x - 2 = x$
Solution set: $\{\quad\}$ or $\emptyset$, since no elements of the universe satisfy the equation

d. $\{x \in N \mid x^2 - 3x = 0\}$
Universe: {natural numbers} or $\{x \mid x \in N\}$
Set selector: $x^2 - 3x = 0$
Solution set: $\{3\}$, since 3 is the only natural number that will satisfy the equation; though 0 satisfies the equation, $0 \notin N$

e. $\{* \in S \mid * \text{ is a vowel}\}$ where S = set of letters of the English alphabet
Universe: Set of letters of the English alphabet
Set selector: $*$ is a vowel
Solution set: $\{a,e,i,o,u\}$

f. $\{x \in N \mid x < 7\}$
Universe: {natural numbers}
Set selector: $x < 7$
Solution set: $\{1,2,3,4,5,6\}$

g. $\{x \in I \mid x^2 = 0\}$
Universe: {integers}
Set selector: $x^2 = 0$
Solution set: $\{0\}$

h. $\{x \in I \mid x^2 < 0\}$
Universe: $\{x \mid x \in I\}$
Set selector: $x^2 < 0$
Solution set: $\emptyset$ or $\{\quad\}$

Exercise 6

1. Describe each of the following sets by the tabulation method or the defining-property method. Suggest another set that contains the given set as a subset.

a. The days of the week beginning with the letter T

b. The capitals of all the states east of the Mississippi River

c. The natural numbers between 4 and 7

d. The rational numbers between 1 and 4

e. The replacements for x from the set of integers that make the sentence

$$x - 5 < -2$$

true

2. Using the symbol $\subset$ or $\not\subset$, write all the relations connecting the following sets in pairs:

$$\begin{aligned} A &= \{\text{quadrilaterals}\} \\ B &= \{\text{rectangles}\} \\ C &= \{\text{squares}\} \\ E &= \{\text{parallelograms}\} \\ F &= \{\text{trapezoids}\} \end{aligned}$$

(*Hint:* There are 20 such cases; see Example 2 in Section 1.7.)

3. Given the following sets, determine which are proper subsets of each other; see Example 3 in Section 1.7.

$$\begin{aligned} R_e &= \text{set of real numbers} \\ N &= \text{set of natural numbers} \\ F &= \text{set of rational numbers} \\ R_e^- &= \text{set of negative real numbers} \\ I^+ &= \text{set of positive integers} \\ R_i &= \text{set of irrational numbers} \\ I &= \text{set of integers} \\ H &= \text{set of squares of negative integers} \end{aligned}$$

4. Which of the following statements are true?

a. $\{\text{dogs}\} \subset \{\text{animals}\}$

b. $\{\text{robins}\} \subset \{\text{birds}\}$

c. $\{\text{people}\} \subset \{\text{women}\}$

d. $\{\text{hexagons}\} \subset \{\text{polygons}\}$

e. $\{\text{teachers}\} \not\subset \{\text{men}\}$

f. $\{\text{tea drinkers}\} \subset \{\text{people}\}$

g. $\{\text{animals}\} \subset \{\text{dogs}\}$

h. $\{\text{women}\} \not\subset \{\text{people}\}$

i. $\{\text{men}\} \subset \{\text{teachers}\}$

j. $\{\text{girls}\} \subset \{\text{females}\}$

k. $\{\text{tigers}\} \not\subset \{\text{cats}\}$

l. $\{\text{women}\} \subset \{\text{teachers}\}$

5. The universe of the variable x in each of the following conditions is the set of integers. Which values from this universe satisfy the given condition?

a. $2x - 7 = 3x - 17$

b. $\frac{3}{4}x + 5 = -1$

c. $2x = -3$

d. $3(2x - 1) = 6x - 3$

e. $2x - 1 = 2x + 1$

f. $x^2 - 7x - 8 = 0$

g. $2x^2 - 5x - 3 = 0$

h. $3x^2 - x = 0$

i. $(3x - 1)(x + 2)(x - \sqrt{3}) = 0$

6. If the universe of the variable x is the set of real numbers, describe the subset of R_e that is the solution set of the given defining condition.

a. $3x = 9$

b. $2x + 4 = 2(x + 2)$

c. $\dfrac{3}{x} = 0$

d. $x^2 = 3$ *e.* $6(x + 2) = x(x + 2)$ *f.* $(2x - 1)(x - 3) = 0$
g. $x^2 + 1 = 0$ *h.* $x^2 < 0$ *i.* $x^2 > 0$
j. $x + 2 = x$ *k.* $5 \cdot x = x \cdot 5$ *l.* $(x^2 + 1)(x - 2) = 0$
m. $x^3 = 0$ *n.* $(x - 1)(x - 3) = 3$

7. Let A be the set of letters $\{\alpha,\beta,\gamma\}$. Find all possible subsets of A. (Recall that the null set is a subset of every set.)

8. Discuss the following:
a. If $a \in A$, $A \subset B$, and $B \subset C$, determine whether $a \in C$.
b. If $a \in A$ and $A \subset B$, does it follow that $a \in B$?
c. If $a \in A$ and $A \in B$, does it follow that $a \in B$?
d. If $a \in A$ and $a \in B$, does it follow that $A \subset B$?
e. If $A \subset B$ and $B \subset D$, does it follow that $A \subset D$?
f. If $a \in B$, $C \subset B$, and $C \subset D$, determine whether $a \in D$.

1.8 NUMBER OF SUBSETS OF A SET

There are occasions when it is necessary to study the set of all subsets of a given finite set. The set of all subsets of a set A is called the power set of A. The notation for the power set of A is 2^A, and thus

$$2^A = \{X \mid X \subseteq A\}$$

For example, Table 1 illustrates the number of subsets that can be formed from a given set A containing 0, 1, 2, or 3 elements. $n(A)$ symbolizes the number of elements in set A and $n(2^A)$ denotes the number of subsets in the power set 2^A.

Table 1

Set A	$n(A)$	2^A	$n(2^A)$
$\emptyset$	0	$\emptyset$	$1 = 2^0$
$\{a\}$	1	$\emptyset$, $\{a\}$	$2 = 2^1$
$\{a,b\}$	2	$\emptyset$, $\{a\}$, $\{b\}$, $\{a,b\}$	$4 = 2^2$
$\{a,b,c\}$	3	$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{b,c\}$, $\{a,c\}$, $\{a,b,c\}$	$8 = 2^3$

If $A = \{a,b,c,d\}$, then it is possible to form subsets containing one element, two elements, three elements, four elements, and no elements. Thus, we have the following subsets: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{a,b,c,d\}$. In all we have listed a total of 16 subsets which include the null set $\emptyset$ and the full set of four elements $\{a,b,c,d\}$.

Since A contained four elements, the power set 2^A was made up of 2^4 or 16 subsets, each subset an element of the power set. Hence we may write

$$2^A = \{\emptyset,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}.$$

It follows that $\{a,b,c\} \in 2^A$, since $\{a,b,c\}$ is one of the elements of 2^A. It is correct to state that $\{\{a,b,c\}\} \subseteq 2^A$. Thus $\emptyset \notin A$, but $\emptyset \in 2^A$; $a \in A$ but $a \notin 2^A$; $\{a,b\} \notin A$, but $\{a,b\} \in 2^A$; A is an element of 2^A, but A is not a subset of 2^A; $\{A\}$ is not an element of 2^A but is a subset of 2^A. We should keep in mind the distinction between the symbols $\in$ and $\subseteq$. One symbol cannot be interchanged for the other, since $\in$ establishes a relation between an element and a set, while $\subseteq$ establishes a relation between one set and another. A careful study of the following examples should make this concept clear.

Example 1. $\{\{2,4\},\{3\},\{0,5\},3\}$ is a set with the four members $\{2,4\}$, $\{3\}$, $\{0,5\}$, and 3. Two of the subsets which can be formed from these elements are $\{\{3\}\}$ and $\{3\}$. $\{\{3\}\} \neq \{3\}$, since the first subset contains the element $\{3\}$ while the second subset contains the element 3.

Example 2. *a.* $\{2\} \in \{\{2\}\}$. This statement is correct, since $\{2\}$ is an element of $\{\{2\}\}$.

b. $\{2\} \subseteq \{\{2\}\}$. This statement is incorrect, since the only element of $\{\{2\}\}$ is $\{2\}$, while the only element of $\{2\}$ is 2.

c. $\{2\} \in \{\{2\},2\}$. This statement is correct, since $\{2\}$ is an element of $\{\{2\},2\}$.

d. $\{2\} \subseteq \{\{2\},2\}$. This statement is correct, since both $\{2\}$ and $\{\{2\},2\}$ contain the number 2 as an element.

When a set A of n elements $x_1, x_2, x_3, \ldots, x_n$ is given and it is desired to form the subsets of 2^A, a decision must be made with respect to each element as to whether it should be included or excluded from the particular subset being formed. Table 2 illustrates these decisions for the various subsets formed from the set of elements a, b, c. Since there are three elements, the formation of each subset involves three decisions. Each element must be considered on the basis of either including or

Table 2

Subset	Elements			Listing
	a	b	c	
A_1	Include	Include	Include	$\{a,b,c\}$
A_2	Include	Include	Exclude	$\{a,b\}$
A_3	Include	Exclude	Include	$\{a,c\}$
A_4	Exclude	Include	Include	$\{b,c\}$
A_5	Include	Exclude	Exclude	$\{a\}$
A_6	Exclude	Include	Exclude	$\{b\}$
A_7	Exclude	Exclude	Include	$\{c\}$
A_8	Exclude	Exclude	Exclude	$\{\ \}$

excluding that element from the particular subset being considered. This means that there are two ways to make a decision with respect to each element and a total of 2^3 or 8 ways to make decisions with respect to all three elements. As a result, there are as many subsets of A as there are ways of making these three decisions. If this argument is extended to a set A containing n elements, there are 2^n ways of making all n decisions and, as a result, 2^n subsets of A. The power set 2^A has 2^n subsets including the null set and A itself.

Exercise 7

1. Write all the subsets for each of the given sets. Indicate which of these subsets are proper subsets. What is the total number of subsets in each case?

a. {Ruth, Elsie}
b. $\{a,b,c\}$
c. $\{A\}$
d. $\{\alpha,\theta,\beta,\sigma\}$
e. $\{1,2,\{1\},\{1,2\},\{2\}\}$
f. $\{0,\{0\},\emptyset,\{\emptyset\}\}$

2. The symbol $n!$ means $n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$, and by definition $0! = 1$. For example, $3!$ means $3\cdot 2\cdot 1$ while $8!$ means $8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$. The symbol $\binom{5}{3} = \frac{5!}{3!2!}$ is used to refer to the number of distinct combinations of five distinct things taken three at a time. For n distinct things taken r at a time,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

These ideas may be used to determine either the total number of subsets which can be formed from n distinct elements or the number of subsets containing exactly k elements where $k = 0, 1, 2, \ldots, n$. If we are given the set $\{1,2,3,4\}$, then the total number of subsets is obtained by the summation $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$, where $\binom{4}{0}$ represents the number of subsets that can be formed containing zero elements, $\binom{4}{1}$ represents the number of subsets that can be formed each containing exactly one element, $\binom{4}{2}$ represents the number of subsets that can be formed each containing two elements, etc. Hence

$$\begin{aligned}\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} &= \frac{4!}{0!\,4!} + \frac{4!}{1!\,3!} + \frac{4!}{2!\,2!} + \frac{4!}{3!\,1!} + \frac{4!}{4!\,0!}\\ &= 1 + 4 + 6 + 4 + 1\\ &= 16\end{aligned}$$

Thus the total number of subsets that can be formed from a set of n distinct elements is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$. (The basis for this conclusion is the expansion of $(1+x)^n$, with $x = 1$, by the binomial theorem.)

Complete the following table:

Set	Number of subsets each containing k elements where k is:			
	One	Two	Three	Five
$\{a,b,c,d,e,f\}$				
$\{1,2,3,4,5,6,7,8\}$				
$\{0,2,4,6,8,10,12,14\}$				

Determine the total number of subsets that can be formed from each of the given sets.

3. How many nonempty subsets can be formed from a finite set of n elements? How many proper subsets can be formed from a finite set of n elements? Is the set of all nonempty subsets equal to the set of all proper subsets? Why?

4. Determine whether each of the following statements is true or false:

a. $3 = \{3\}$ *b.* $3 \in \{3\}$ *c.* $\emptyset = 0$
d. $0 \in \emptyset$ *e.* $5 \in \{\{5\}\}$ *f.* $3 \in \{3,4\}$
g. $4 \in \{\{4\},4\}$ *h.* $2 \in \{\{1,3\},\{2\}\}$ *i.* $\{3,4\} \subset \{3,4,5\}$
j. $\{3,4\} \subseteq \{\{3,4\},\{5,6\}\}$ *k.* $\{5,6,7\} \subset \{5,6,7\}$ *l.* $\{2,\{8\}\} \subset \{2,8,9\}$
m. $\{5\} \notin \{3,5,\{5\}\}$

5*a*. If $V = \{1,2,3\}$, find the power set 2^V of V.
b. If $T = \{a,b,c,d,e\}$, find 2^T of T.

6. For all sets S, T, and V, determine whether the following statements are true or false. For each of the false statements construct an example which would support your conclusion. (Sets T, S, and V for statement *a* are not necessarily the same as T, S, and V in statement *b*, etc.)
a. If $T = S$ and $S = V$, then $T = V$.
b. If $T \subset V$ and $T \subset S$, then $V \subset S$.
c. If $T \in S$ and $S \in V$, then $T \in V$.
d. If $T = V$ and $T \in S$, then $V \in S$.
e. If $T \subset S$ and $S \in V$, then $T \in V$.

1.9 OPERATIONS ON SETS

New sets can be formed from given sets of a particular universe by combining them in a prescribed manner. The given sets are subsets of some universal set, and the new sets formed are also subsets of the same universe.

If A and B are two subsets of a universal set U, we have the following.

a. The intersection of A and B, written $A \cap B$, is the set of elements which are in both A and B at the same time. Putting it another way, $A \cap B$ is the set of elements common to both A and B. $A \cap B$ is read "A intersection B," or "A cap B," or the "product of A and B,"

or the "meet of A and B." Thus

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$
$$= \{x \mid x \in A \wedge x \in B\}$$

The symbol "$\wedge$" meaning "and" is used to bring together two conditions "$x \in A$" and "$x \in B$" and implies that each element of the set must satisfy the first condition and the second condition simultaneously.

Example 1:

$$\{1,2,3,4\} \cap \{2,3,5\} = \{2,3\}$$
$$\{1,3,5\} \cap \{2,4,6\} = \emptyset$$

b. The union of A and B, written $A \cup B$, is the set of elements which belong to A or to B, or to both A and B. $A \cup B$ is read "A union B," or "A cup B," or the "sum of A and B," or the "join of A and B." Thus

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$
$$= \{x \mid x \in A \vee x \in B\}$$

The symbol "$\vee$" meaning "or" is used to bring together two conditions and implies that each element of the set must satisfy the first condition or the second condition, or both. "Or" is used here in the sense of "and/or" and is referred to as the "inclusive or"; that is, x is a member of either A or B and can be an element of both A and B. If the "exclusive or," written as "V," is used, then x is a member of either A or B but not both A and B.

Example 2:

$$\{1,2,3,4\} \cup \{2,3,5\} = \{1,2,3,4,5\}$$
$$\{3,4,5\} \cup \{2,4,6\} = \{2,3,4,5,6\}$$
$$\{3,4\} \cup \{\quad\} = \{3,4\} \cup \emptyset = \{3,4\}$$

c. The complement of A, written A', represents the set of all the elements of U, the universe, which are not elements of A. Thus

$$A' = \{x \mid x \in U \text{ and } x \notin A\}$$
$$= \{x \mid x \in U \wedge x \notin A\}$$

Example 3. Suppose $U = \{2,4,6,8,10\}$ and $A = \{2,6\}$; then

$$A' = \{4,8,10\}$$

Example 4. If $U = \{2,4,6,8,10,12,14\}$, $A = \{2,4,6\}$, $B = \{2,6,10,14\}$, and $C = \{6,10,14\}$, find A', B', C', $A \cup B$, $A \cup C$, $B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$, $(A')'$, $(B \cap C)'$, $(A \cup B)'$, $(A \cap B) \cup C$, $(A \cup C) \cap (A \cap C)$, U'.

Answer:

$A' = \{8,10,12,14\}$
$B' = \{4,8,12\}$
$C' = \{2,4,8,12\}$
$A \cup B = \{2,4,6,10,14\}$
$A \cup C = \{2,4,6,10,14\}$
$B \cup C = \{2,6,10,14\} = B$
$A \cap B = \{2,6\}$
$A \cap C = \{6\}$
$B \cap C = \{6,10,14\} = C$
$(A')' = A = \{2,4,6\}$
$(A \cup B)' = \{8,12\}$
$(B \cap C)' = \{2,4,8,12\}$
$(A \cap B) \cup C = \{2,6\} \cup \{6,10,14\} = \{2,6,10,14\}$
$(A \cup C) \cap (A \cap C) = \{2,4,6,10,14\} \cap \{6\} = \{6\}$
$U' = \{\quad\} = \emptyset$

Exercise 8

1. If $A = \{\text{John, Henry, Mary, Jean}\}$ and $B = \{\text{Mary, Jean, Ruth, Elsie}\}$, find $A \cap B$ and $A \cup B$.

2*a*. Find the complement of the set of odd natural numbers where the universe is the set of natural numbers.

b. Find the complement of the set of negative integers where the universe is the set of integers. This new set is referred to as the set of nonnegative integers. Why is it not called simply the set of positive integers?

3. Let $U = \{1,2,3,4,5,6,7,8\}$, $A = \{1,2,3,4\}$, and $B = \{4,6,8\}$. Find A', B', $A \cap B$, $A \cup B$, $A \cap B'$, $A' \cap B$, $A' \cup B$, $A \cup B'$, $A' \cup B'$, $(A \cup B)'$, $(A \cap B)'$, $A' \cap B'$, and $(A' \cup B')'$.

4. Let $U = \{\text{natural numbers}\}$
$A = \{\text{positive odd integers}\} = \{1,3,5, \ldots\}$
$B = \{\text{positive even integers}\} = \{2,4,6, \ldots\}$
$C = \{\text{positive integers that are multiples of } 5\} = \{5,10,15, \ldots\}$

Find:

a. $A \cap B$ *b*. $A \cup B$ *c*. A'
d. B' *e*. $A \cap C$ *f*. $B \cap C$
g. $U \cap A$ *h*. $A \cap B \cap \emptyset$

5. Let $A = \{\text{all points on the straight line } L_1\}$
$B = \{\text{all points on the straight line } L_2\}$

Describe $A \cap B$ for each of the following conditions:
a. L_1 is parallel to L_2 but not coincident with it.
b. L_1 coincides with L_2.
c. L_1 and L_2 are two nonparallel lines.

6. Describe $A \cap B$ and $A \cup B$ for each of the following conditions:
a. Sets A and B have no elements in common.
b. Sets A and B are equal.
c. Set A is a subset of B.
d. Set B is a subset of A.

7. In the diagram shown in Fig. 3, U is the set of all points in rectangle $ABEF$, L is the set of points in triangle ADF, M is the set of points in trapezoid $ADEF$, T

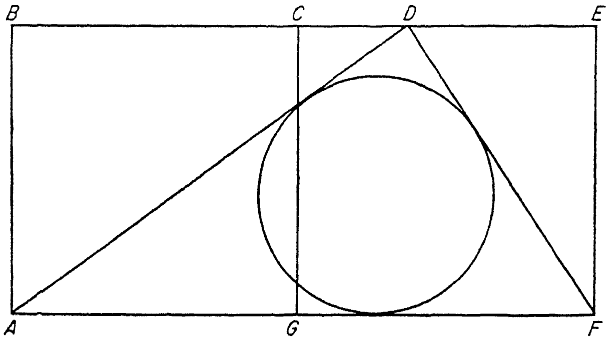


FIG. 3

is the set of points in the circle, and W is the set of points inside the square $ABCG$. Make copies of this diagram and shade in the set described by each of the following expressions:

a. $L \cap M$ *b.* $L \cap T$ *c.* $T \cap L$
d. $M' \cap W'$ *e.* $T \cap W$ *f.* $T' \cap L$
g. $W' \cap T' \cap M'$ *h.* $M' \cap T$

Describe each of the following by using L, M, T, W, $\cap$, $\cup$, and $'$:

i. Set of points in triangle DEF
j. Set of points in triangle ABD
k. Set of points in rectangle $CEFG$
l. Set of points included in the region between the two intersecting lines AD and CG
m. Set of points included in the region bordered by the lines CG and DF
n. Set of points included in the region bordered by line CG on the left and the circle on the right

1.10 VENN DIAGRAMS

Venn diagrams, named for the English logician John Venn (1834–1883), diagrammatically represent the relations of membership and inclusion and the operations of union, intersection, and complementation.

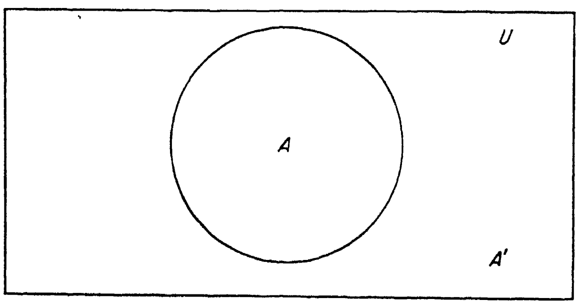


FIG. 4

A rectangle is drawn to represent the universe U. Subsets of U are represented by circles considered as subspaces of the rectangle. This results in the type of representation shown in Fig. 4. The elements of U are represented by the points within the rectangle, the elements of A by the points within the circle, and the elements of A' by the points within that part of the rectangle outside the region representing A. Thus $U = A \cup A'$ and $\emptyset = A \cap A'$.

If we are given two sets A and B that overlap (have some elements in common), two intersecting circles are drawn within the rectangle (Fig. 5).

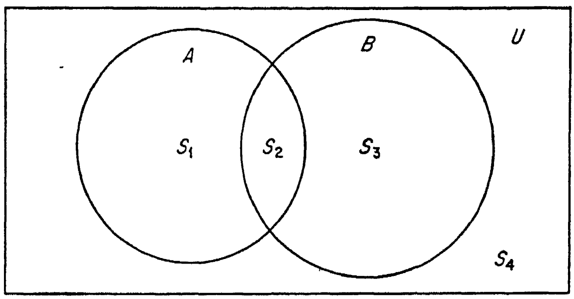


FIG. 5

U is represented by the regions S_1, S_2, S_3, and S_4. These regions can be described in terms of sets A and B as follows:

a. "$x \in A$ and $x \notin B$" is in region S_1 and can be described as set $A \cap B'$. $A \cap B' = \{x \mid x \in A \wedge x \notin B\}$.

b. "$x \in A$ and $x \in B$" is in region S_2 and can be described as set $A \cap B$. $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

c. "$x \notin A$ and $x \in B$" is in region S_3 and can be described as set $A' \cap B$. $A' \cap B = \{x \mid x \notin A \wedge x \in B\}$.

d. "$x \notin A$ and $x \notin B$" is in region S_4 and can be described as set $A' \cap B'$. $A' \cap B' = \{x \mid x \notin A \wedge x \notin B\}$.

Note that regions S_1 through S_4 represent disjoint sets; that is, the intersection of any two regions yields the null set. The set $A \cap B'$ includes those elements in A that are not in B. Here our concern is not with all the elements that are in B' but with a subset of B'. Thus, the set $A \cap B'$ or $B' \cap A$ can be designated as $A - B$, where

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Example 1. If $U = \{1,2,3,4,5,6\}$, $A = \{1,2,5,6\}$, and $B = \{2,4,5\}$, then $A - B = \{1,6\}$ and $B - A = \{4\}$. Since $B' = \{1,3,6\}$ and $A' = \{3,4\}$, we can write $(A - B) \subset B'$ and $(B - A) \subset A'$. Note that $A - B \neq B - A$ except when $A = B$.

Example 2. Sets, such as $(A \cap B)'$, $A \cup B$, $A \cap B$, A', and B', formed through set operations may be studied by the use of the Venn diagram shown in Fig. 6 in the following manner.

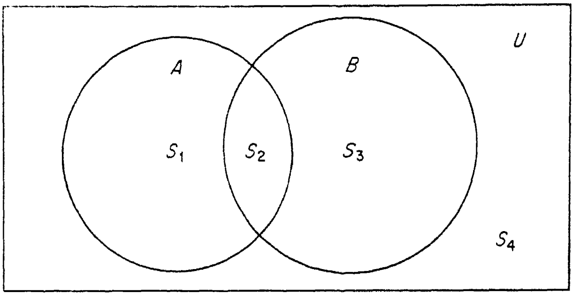


FIG. 6

a. $(A \cap B)'$ is the set of all elements not contained in A and B as common elements. Thus it is represented by the union of regions S_1, S_3, and S_4. The following tabular arrangement could be used where the regions are considered to be elements of each set.

Set	A	B	$A \cap B$	$(A \cap B)'$
Regions	S_1, S_2	S_2, S_3	S_2	S_1, S_3, S_4

b. $A \cup B$ is the set of all elements in either A or B, or both. Thus it is represented by the union of regions S_1, S_2, and S_3.

Set	A	B	$A \cup B$
Regions	S_1, S_2	S_2, S_3	S_1, S_2, S_3

c. $A \cap B$ is the set of all elements in both A and B (common elements) at the same time. Thus it constitutes region S_2.

Set	A	B	$A \cap B$
Regions	S_1, S_2	S_2, S_3	S_2

d. A' constitutes the set of all elements not in A. Thus it is represented by the union of regions S_3 and S_4.

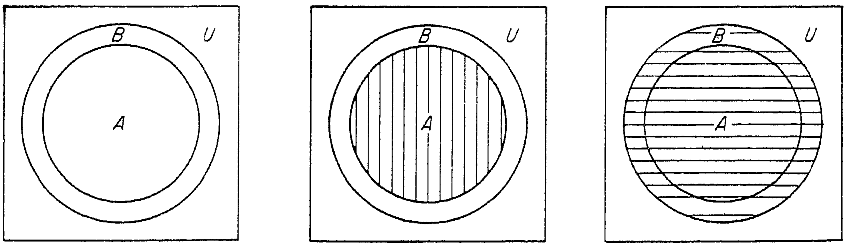
Set	A	A'
Regions	S_1, S_2	S_3, S_4

e. B' constitutes the set of all elements not in B. Thus it is represented by the union of regions S_1 and S_4.

Set	B	B'
Regions	S_2, S_3	S_1, S_4

Example 3. If in Example 2 it is assumed that $A \subseteq B$, then the designated regions S_1, S_2, S_3, and S_4 of Fig. 6 take on the following meanings in terms of sets A and B. If $A \subseteq B$, then every element of A is in B, and it follows that $S_1 = \emptyset$. Since $S_1 = A \cap B'$, then $A \cap B' = \emptyset$, and the regions S_1 and S_2 are identical with the one region S_2. The interpretation would be that $A \cap B = A$. Correspondingly, S_1, S_2, and S_3 are identical with the two regions S_2 and S_3 and represent the set $A \cup B$. In summary, if we imply that every element of A is also an element of B, then $A \subseteq B$, $A \cap B' = \emptyset$, $A = A \cap B$, and $B = A \cup B$.

If $A \subset B$, it is customary to draw the circle representing A entirely within the circle representing B, as shown in Fig. 7.



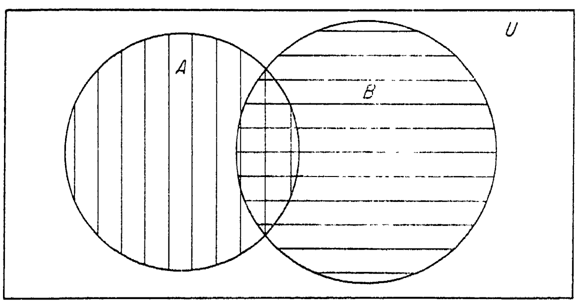
$A \subset B$
(*a*)

$A \cap B = A$
(*b*)

$A \cup B = B$
(*c*)

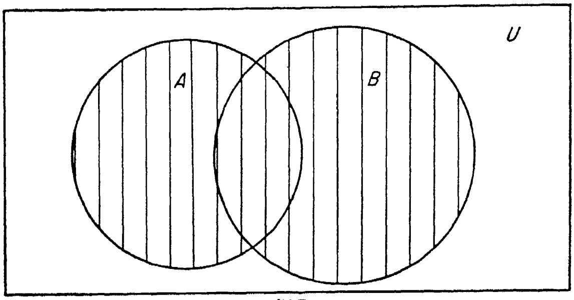
FIG. 7

Shading or cross-hatching is also used to indicate the region under discussion. The following example will clarify this procedure. To illustrate the set $A \cap B$ we use vertical shading within circle A for the elements of set A and horizontal shading within circle B for the elements of set B. The region containing both vertical and horizontal shading represents $A \cap B$, as shown in Fig. 8. To represent $A \cup B$ only one type of shading is necessary, as shown in Fig. 9.



$A \cap B$

FIG. 8



$A \cup B$

FIG. 9

Example 4. Two sets are "disjoint or mutually exclusive" if their intersection is the null set $\emptyset$; that is, the two sets have no common elements. Thus, if A and B are disjoint sets relative to U, $A \cap B = \emptyset$, as shown in Fig. 10.

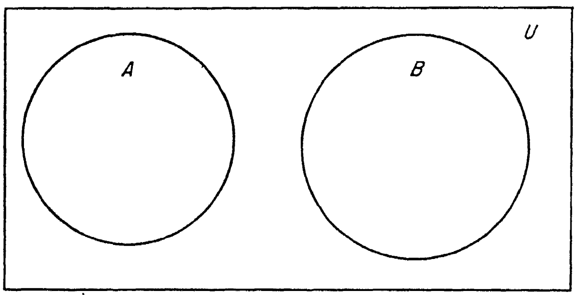


FIG. 10

Example 5. If A, B, and C are subsets of a universal set U, then a Venn diagram (Fig. 11) may be drawn in which three circles and eight regions are exposed. We describe these regions in terms of sets A, B, and C:

Region	*Description*
S_1	$\{x \mid x \in A \wedge x \in B \wedge x \in C\} = (A \cap B) \cap C$
S_2	$\{x \mid x \in A \wedge x \in B \wedge x \notin C\} = (A \cap B) \cap C'$
S_3	$\{x \mid x \in A \wedge x \notin B \wedge x \in C\} = (A \cap B') \cap C$
S_4	$\{x \mid x \in A \wedge x \notin B \wedge x \notin C\} = (A \cap B') \cap C'$
S_5	$\{x \mid x \notin A \wedge x \in B \wedge x \in C\} = (A' \cap B) \cap C$
S_6	$\{x \mid x \notin A \wedge x \in B \wedge x \notin C\} = (A' \cap B) \cap C'$
S_7	$\{x \mid x \notin A \wedge x \notin B \wedge x \in C\} = (A' \cap B') \cap C$
S_8	$\{x \mid x \notin A \wedge x \notin B \wedge x \notin C\} = (A' \cap B') \cap C'$

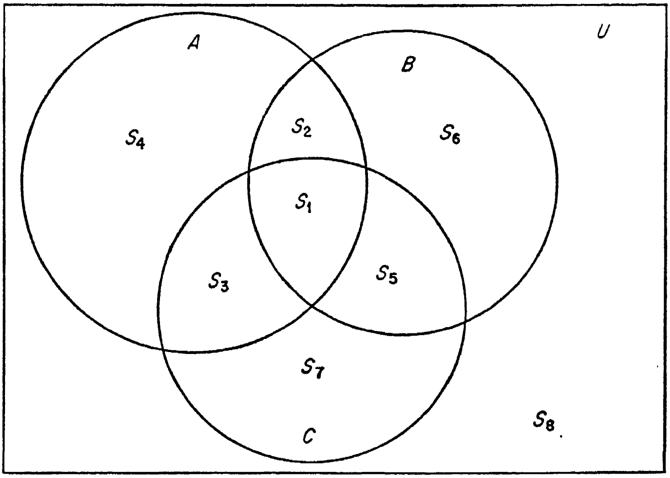


FIG. 11

Note that regions S_1 through S_8 represent disjoint sets; i.e., the intersection of any two regions yields the null set.

Example 6. Each of the sets U, A, C, $A \cap B$, $A \cup B$, $(A \cup B) \cap C$, A', $A' \cap (A \cap B)$, $B \cap C$, and $(A \cap B') \cap C'$ may be studied with respect to Fig. 11:

	Set	*Regions*
a.	U	S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8
b.	A	S_1, S_2, S_3, S_4
c.	C	S_1, S_3, S_5, S_7
d.	$A \cap B$	S_1, S_2
e.	$A \cup B$	S_1, S_2, S_3, S_4, S_5, S_6
f.	$(A \cup B) \cap C$	S_1, S_3, S_5
g.	A'	S_5, S_6, S_7, S_8
h.	$A' \cap (A \cap B) = \emptyset$	Empty space
i.	$B \cap C$	S_1, S_5
j.	$(A \cap B') \cap C'$	S_4

Example 7. By using the relationships of set equality and set inclusion, the following sets may be written in a sequential order: $A \cup B$, $U, \emptyset, A \cap B, A \cup (B \cup C), B, (A \cap B) \cap C, (A \cup B) \cup C, B \cap A, \emptyset'$.

Sequence:

$$\emptyset \subseteq (A \cap B) \cap C \subseteq B \cap A$$
$$= A \cap B \subseteq B \subseteq A \cup B \subseteq (A \cup B) \cup C = A \cup (B \cup C) \subseteq U = \emptyset'$$

This can be verified by the use of a Venn diagram such as that shown in Fig. 11.

Example 8. The sets $A \cup B'$ and $(A' \cap B)'$ are identical. This is illustrated by the Venn diagram shown in Fig. 12 and by Table 1. Since

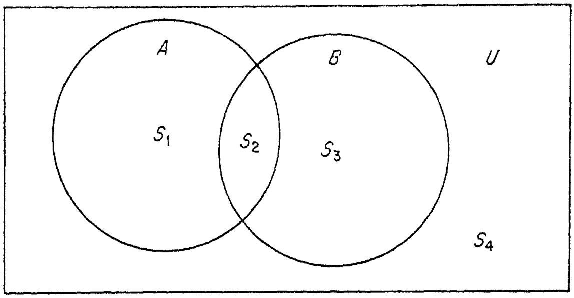


FIG. 12

Table 1

$A \cup B'$		$(A' \cap B)'$	
Set	Regions	Set	Regions
A	S_1, S_2	A'	S_3, S_4
B'	S_1, S_4	B	S_2, S_3
$A \cup B'$	S_1, S_2, S_4	$A' \cap B$	S_3
		$(A' \cap B)'$	S_1, S_2, S_4

both $A \cup B'$ and $(A' \cap B)'$ contain the same regions, we have demonstrated the equality of these sets. Individual Venn diagrams for the sets $A \cup B'$ and $(A' \cap B)'$, with the pertinent regions shaded, will also verify the equality of these sets.

1.11 THE NUMBER OF ELEMENTS IN SET A

If A is any set, then $n(A)$ represents the number of elements in A. Hence if A and B are disjoint ($A \cap B = \emptyset$), then the number of elements in the union of A and B is equal to the sum of the number of elements in A and the number of elements in B. Consequently,

$$n(A \cup B) = n(A) + n(B)$$

If A and B are not necessarily disjoint, then $n(A \cup B)$ is obtained in the following manner. First, we know that $B \cap B' = \emptyset$, since B' is the complement of B and as a result $A \cap B$ and $A \cap B'$ are disjoint sets. Thus,

$$n[(A \cap B) \cup (A \cap B')] = n(A \cap B) + n(A \cap B')$$

Similarly, since $A \cap A' = \emptyset$, $A \cap B$ and $A' \cap B$ are disjoint sets and

$$n[(A \cap B) \cup (A' \cap B)] = n(A \cap B) + n(A' \cap B)$$

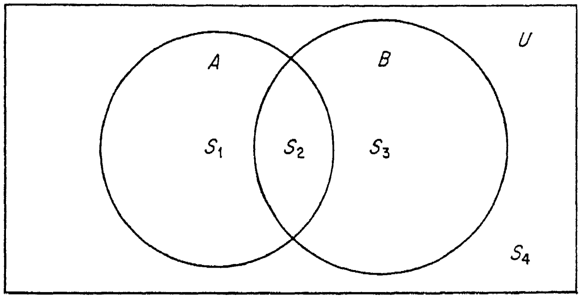


FIG. 13

By the use of Fig. 13, it can now be shown that

$$(A \cap B) \cup (A \cap B') = A$$

and

$$(A \cap B) \cup (A' \cap B) = B$$

As a consequence,

$$n(A \cap B) + n(A \cap B') = n(A)$$

and

$$n(A \cap B) + n(A' \cap B) = n(B)$$

Adding the left members and the right members, respectively, of these two equations and then subtracting $n(A \cap B)$ from the results, we obtain

$$n(A \cap B) + n(A \cap B') + n(A' \cap B) = n(A) + n(B) - n(A \cap B)$$

If, once more, Fig. 13 is examined, we find that the three sets $A \cap B'$, $A \cap B$, and $A' \cap B$ are disjoint sets. Set $A \cap B'$ is represented by S_1, $A \cap B$ by S_2, and $A' \cap B$ by S_3. The number of elements in S_1, S_2, and S_3 is the same as the number of elements in $A \cup B$. Hence,

$$n(A \cup B) = n(A \cap B') + n(A \cap B) + n(A' \cap B)$$

It now follows that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, if we wish to determine the number of elements in the set $A \cup B$ where $A \cap B \neq \emptyset$, we determine the sum of the number of elements in A and the number of elements in B and from the result subtract the number of elements in the intersection of A and B. This subtraction is necessary because the elements in $A \cap B$ have been counted twice.

Example 1. Suppose we are informed by the registrar of our school that 600 students take mathematics and 300 take physics. How many different students are enrolled in the two courses?

If M is the set of students enrolled in mathematics and P the set of physics students, then we desire to find $n(M \cup P)$. But

$$n(M \cup P) = n(M) + n(P) - n(M \cap P)$$

$n(M \cap P)$ represents the number of students enrolled in both mathematics and physics. Since this is not given, we cannot answer the question. However, if we were told that 173 students are enrolled in both subjects, then $n(M \cup P) = 600 + 300 - 173 = 727$. Therefore 727 different students are enrolled in the two courses.

Example 2. The school newspaper reports that the combined membership of the Mathematics Club and the Chemistry Club is 122 students. What is the total membership of the Chemistry Club if 50 students are known to be members of the Mathematics Club and 28 students are members of both organizations?

$$\begin{aligned} n(M \cup C) &= n(M) + n(C) - n(M \cap C) \\ 122 &= 50 + n(C) - 28 \\ n(C) &= 100 \end{aligned}$$

The Chemistry Club has 100 members.

Example 3. In a certain high school 60 per cent of the students purchased tickets to a dance, and 70 per cent purchased tickets to a football game. At least how many purchased tickets to both events? Let $n(A) = 60$ per cent, $n(B) = 70$ per cent, and $n(A \cup B) = 100$ per cent. Then

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 100 &= 60 + 70 - n(A \cap B) \\ n(A \cap B) &= 30 \end{aligned}$$

At least 30 per cent of the students purchased tickets to both events.

Example 4. For any three sets A, B, and C,

$$\begin{aligned} n(A \cup B \cup C) = n(A) + n(B) + n(C) &- n(A \cap B) \\ - n(A \cap C) - n(B \cap C) &+ n(A \cap B \cap C) \end{aligned}$$

A Venn diagram may be used to illustrate this theorem. The extension of this theorem to more than three sets results in formulas that are very cumbersome.

We illustrate this formula with the following example:

In the graduating class of 200 students of a certain high school, records indicate that 80 students have taken physics, 90 have taken biology,

55 have taken chemistry, 32 have taken both biology and physics, 23 have taken both chemistry and physics, 16 have taken both biology and chemistry, and 8 have taken all three subjects. Are the records accurate? We are assuming that each of the 200 students was enrolled in at least one of the three courses. If

$$B = \text{set of students who have taken biology}$$
$$C = \text{set of students who have taken chemistry}$$
$$P = \text{set of students who have taken physics}$$

then

$$n(B) = 90 \qquad n(C) = 55 \qquad n(P) = 80$$
$$n(B \cap C) = 16 \qquad n(B \cap P) = 32 \qquad n(C \cap P) = 23$$
$$n(B \cap C \cap P) = 8$$

Hence

$$\begin{aligned} n(B \cup C \cup P) &= 90 + 55 + 80 - 16 - 32 - 23 + 8 \\ &= 162 \end{aligned}$$

The information is inconsistent, since we were initially told that there was a total of 200 students. With the information that the records were absolutely correct and without the assumption that all students had taken at least one of the subjects, we could now conclude that 38 students had not taken any of the three courses. Thus $n(P' \cap B' \cap C') = 38$. This conclusion is illustrated in the Venn diagram shown in Fig. 14.

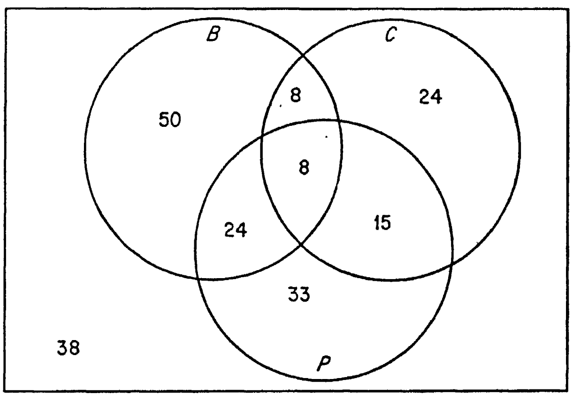


FIG. 14

In studying problems of this type the analysis should begin with the set $P \cap B \cap C$ and then extend outwardly in all directions. This enables us to examine first those elements common to all three sets, then those elements common to two of the sets, and finally those elements that appear in each set that are not contained in any of the others.

1.12 LAWS OF OPERATIONS

It is important to note that the operations of union, intersection, and complementation possess certain specified properties regardless of the selection of the universe U and the designation of the nonempty subsets of U. These three defined operations are applied to sets and generate other sets to which the same operations can be applied. The operations on sets are governed by the following laws or postulates.

If A, B, C, . . . are subsets of some universal set U, then:

Commutative Laws

1a. $A \cup B = B \cup A$ 1b. $A \cap B = B \cap A$

Associative Laws

2a. $(A \cup B) \cup C = A \cup (B \cup C)$ 2b. $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws

3a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 3b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Identity Laws

4a. $A \cup \emptyset = A$ 4b. $A \cap U = A$

Complement Laws

5a. $A \cup A' = U$ 5b. $A \cap A' = \emptyset$

6. $(A')' = A$

These laws may be verified through the use of Venn diagrams in a manner similar to that used in Example 8 of Section 1.10. In Chapter 5 we shall consider these postulates with respect to a mathematical structure.

Exercise 9

1. Let I = set of integers
   I^+ = set of positive integers
   I^- = set of negative integers
   P = set of primes
   A = set of positive even integers
   B = set of positive odd integers
   G = set of positive integers that are multiples of 3

Complete the following table by filling in each blank with one of the statements $S \cap T = \emptyset$, $S \cap T \neq \emptyset$, $S \subset T$, $S \supset T$, or $S = T$, where S represents any of the given sets in the left column and T represents any of the given sets in the top row.

	I^+	I^-	P	A	B	G	I
I^+	$I^+ = I^+$						$I^+ \subset I$
I^-							
P				$P \cap A \neq \emptyset$			
A					$A \cap B = \emptyset$		
B							
G							
I							

2. Let U = set of positive integers less than 20
A = set of positive even integers less than 20
B = set of positive even integers less than 17 that are multiples of 4

Find:

a. $A - B$ *b.* $B - A$ *c.* $(A - B)'$
d. $(B - A)'$ *e.* $(A - B) \cup (B - A)$ *f.* $(A - B) \cap (B - A)$
g. $(A - B)' \cup (B - A)'$ *h.* $(B - A)' \cap (A - B)'$

3. Verify each of the following postulates by the use of Venn diagrams. Follow the procedure suggested in Example 8, Section 1.10.

a. $A \cup B = B \cup A$ *b.* $A \cap B = B \cap A$
c. $(A \cup B) \cup C = A \cup (B \cup C)$ *d.* $(A \cap B) \cap C = A \cap (B \cap C)$
e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
f. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. If A, B, and C are three nonempty distinct subsets of some universe U, draw a Venn diagram to verify that all the statements in each of the following cases are true. Part *a* is illustrated in Fig. 15.

a. $A \subset B, B \cap C \neq \emptyset, A \cap C = \emptyset, C \not\subset B$
Answer:

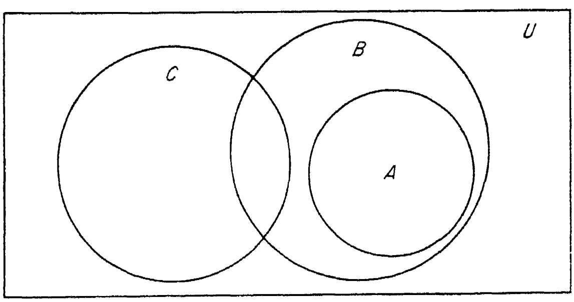


FIG. 15

b. $A \cap B \neq \emptyset, A \cap C = \emptyset, B \cap C \neq \emptyset$
c. $A \cap B = \emptyset, A \cap C = \emptyset, B \cap C \neq \emptyset$
d. $A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset$
e. $A \cap B = \emptyset, A \subset C, B \subset C$
f. $A \subset B, B \subset C$
g. $A \cap B \cap C \neq \emptyset, A \not\subset B, B \not\subset A, C \not\subset (A \cup B), (A \cup B) \not\subset C$

5. Specify what the following sets represent for each of the Venn diagrams drawn for the parts of Problem 4.

a. $A \cap (B \cap C)$ *b.* $(A \cap B) \cup C$ *c.* $(A \cup B) \cap C$
d. $A \cup (B \cap C)$ *e.* $A \cap (B \cup C)$

6. Using the Venn diagram shown in Fig. 16, determine the region or combination of regions represented by each of the following:

a. $(A \cup B)'$ *b.* $A' \cup B'$ *c.* $A \cup B$
d. $A \cap B$ *e.* $A \cup B'$ *f.* $(A \cup B) \cap B'$
g. $A' \cap B'$ *h.* $A' \cap B$ *i.* $A' \cap (A \cup B')$
j. $(A' \cap B')'$ *k.* $(A' \cup B)' \cap A'$ *l.* $[(A' \cap B)' \cup A]'$

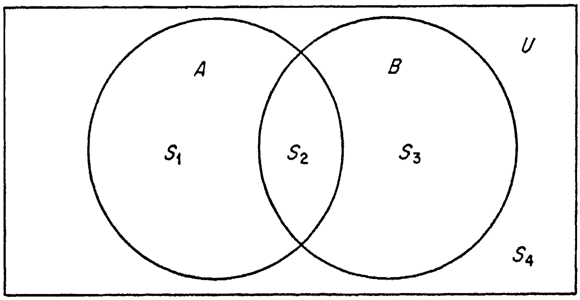


FIG. 16

7. For the parts of Problem 6, find those which represent the same region or combination of regions.

8. Homeroom A of a certain high school has 18 students taking physics, 23 students taking biology, and 24 students taking history. Of these students 14 are in both biology and history, 12 in physics and biology, 11 in history and physics, and 6 in all three subjects. Each student in homeroom A is taking at least one of the three subjects.

a. How many students are in homeroom A?
b. How many students are taking history but not biology?
c. How many students are taking history but not biology or physics?
d. How many students are taking exactly one of the three courses?

9. In the city of Utopia there are 1000 families. A survey indicated that 470 subscribe to *Life*, 420 subscribe to *Digest*, and 315 subscribe to *Post*. Of these subscribers 140 take both *Post* and *Digest*, 220 take both *Post* and *Life*, 110 take both *Digest* and *Life*, and 75 take all three.

a. How many families do not subscribe to any of these periodicals?
b. How many families subscribe to exactly one of these periodicals?
c. How many families subscribe to exactly two periodicals?

10. At a picnic attended by 50 children, 21 children participated in the pie-eating contest, 20 participated in the baseball toss, and 25 participated in the sack race. Of these, seven participated in both the sack race and the baseball toss, four participated in both the pie-eating contest and the baseball toss, eight participated in the sack race and the pie-eating contest, and three did not participate in any of the events.

a. How many participated in all three events?

b. How many participated in exactly one event?

c. How many were not able to participate in any of the other events after the pie-eating contest?

11. The information in the following table is the result of a survey conducted in a certain industry.

Classification	*Per cent of employees*
College graduates	65
Male employees	75
Married employees	80
Employees with more than 5 years' service	85

a. At least what per cent are married men?

b. At least what per cent are male employees and have more than 5 years' service?

c. At least what per cent are married and have more than 5 years' service?

d. At least what per cent are male college graduates?

e. At least what per cent are males, college graduates, and have more than 5 years' service?

f. At least what per cent are males, married, and college graduates?

PROJECTS

Supplementary Exercises

1. Denoting the set of real numbers by R_e, the set of rational numbers by F, the set of irrational numbers by R_i, the set of integers by I, and the set of natural numbers by N, determine whether each statement is true or false.

$U = R_e$, the set of real numbers.

a. $N \subset I$ *b.* $R_i \subset F$ *c.* $R_i \cup F = R_e$
d. $R_i \cap F = \emptyset$ *e.* $I \subset N$ *f.* $R_e \subset F$
g. $N \subset F$ *h.* $I \subset R_e$ *i.* $N \cap I = I$
j. $F' = R_i$ *k.* $F \supset I$ *l.* $N \subset I \subset F \subset R_e$

2. List all the subsets for the set $A = \{1,2,4,8,16\}$. Find the sum of the elements in each subset. After examining these sums, state a generalization concerning them and the elements 1, 2, 4, 8, 16. (*Hint:* Arrange sums obtained in increasing order.)

3. A set of subsets of S is said to be disjoint if no two of the subsets have an element in common. Consider the set 2^V where $V = \{1,2,3\}$. Determine all the possible sets of subsets of V that are disjoint.

4. Using the Venn diagram shown in Fig. 17, determine the region or combination of regions represented by each of the following:

a. $(A \cup B) \cup C$ *b.* $(A \cup B) \cap C$
c. $(A \cap B) \cup C$ *d.* $(A \cap B) \cup C'$
e. $(A' \cap B') \cup C'$ *f.* $(A \cap B)' \cap C$
g. $(A \cup C) \cap (B \cup C)$ *h.* $(A \cap B)' \cup (A \cap C)'$
i. $(A' \cap C) \cup (B' \cap C)$ *j.* $(A \cup C') \cap (B \cup C')$
k. $(A' \cap B')' \cup A'$ *l.* $B \cup [(A \cup B) \cup C]'$

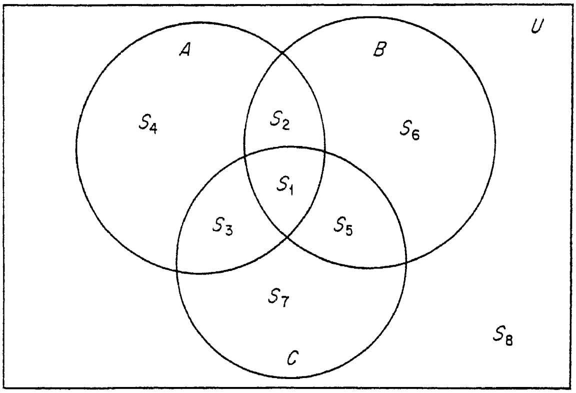


FIG. 17

5. A and B are subsets of U. Determine which of the following statements are false. For each false statement construct an example of particular sets A and B which shows that the statement is in general not true.

a. $A \subseteq A \cup B$ *b.* $B \subseteq A \cap B$ *c.* $A \cap B \subseteq B$
d. $A \cap B \subseteq A \cup B$ *e.* $(A \cap B) \cup A = A$ *f.* $A \cup B \supset A \cap B$
g. $B' \subset (A \cap B)'$ *h.* $(A \cup B)' \subseteq A'$ *i.* $A \cup (A \cap B)' = U$
j. $(A \cup B)' \subseteq (A \cap B)'$

6. Complete each of the following:

a. $\emptyset \cap U =$ *b.* $A \cap \emptyset =$ *c.* $A \cup U =$
d. $\emptyset \cap \emptyset =$ *e.* $A \cup A' =$ *f.* $A \cap A' =$
g. $\emptyset \cup U =$ *h.* $A \cup \emptyset =$ *i.* $\emptyset \cap \{\emptyset\} =$
j. $\{\emptyset\} \cap \{\emptyset\} =$

7. Let $U = \{1,2,\{1\},\{2\},\{1,2\}\}$
$A = \{1\}$ $C = \{1,2,\{1\}\}$
$B = \{1,2\}$ $D = \{\{1\},\{2\}\}$
$E = \{1,2,\{1,2\}\}$

Find:

a. $A \cup B$ *b.* $A \cap B$ *c.* $A \cup C$
d. $A \cap D$ *e.* $A \cap C$ *f.* $B \cap C$
g. $D' \cap E$ *h.* $D \cap E$ *i.* $D \cap C$
j. $(A \cap C) \cap D$ *k.* $A' \cap C'$ *l.* $(D \cap E') \cup A$
m. $(C' \cap E')' \cap D$ *n.* $(D \cap E)' \cup (A \cap D)$

8. Draw a Venn diagram and show that the sets in each of the following expressions are identical:

a. $(A \cup B)'$ and $A' \cap B'$ *b.* $(A \cap B)'$ and $A' \cup B'$
c. $(A \cup B) \cup C$ and $A \cup (B \cup C)$
d. $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$
e. $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$

9*a.* Verify the following:
(1) $U' = \emptyset$ and $\emptyset' = U$.
(2) If $A \subseteq B$, then $B' \subseteq A'$.
b. Suppose $A \cup B = \emptyset$. What conclusions do you draw about sets A and B?

10. Given: U = set of the 52 cards in a deck of playing cards
S = subset of spades
D = subset of diamonds
C = subset of clubs
H = subset of hearts
K = subset of cards that are honor cards, i.e., tens, jacks, queens, kings, and aces

a. Identify each of the following sets and determine the number of elements, $n(A)$, in each.

(1) $S \cap K$ (2) K' (3) $D \cap S$
(4) $D \cap S'$ (5) $D \cup S \cup K$ (6) $(S \cup D) \cap K$

b. Represent the following statements in symbolic form:
(1) The set of cards that are not honor cards
(2) The set of cards that are neither spades nor honor cards
(3) The set of clubs or hearts that are not honor cards
(4) The set of cards that are neither hearts nor honor cards
(5) The set of hearts or diamonds that are honor cards

11. A recent survey of 200 students majoring in science revealed that the number studying one or more of the subjects mathematics (M), physics (P), or chemistry (C) is as follows:

Subject	*Number of students*
M	100
P	70
C	46
M and P	30
M and C	28
P and C	23
M, P, and C	18

a. How many students were not enrolled in any of these subjects?

b. How many students had mathematics as their only subject? Physics? Chemistry?

c. How many students had mathematics and physics as their only subjects? Mathematics and chemistry? Physics and chemistry?

12. A recent questionnaire addressed to 353 public school teachers requested answers to the following two questions:

a. How long have you been teaching?

b. Would you advocate a 12-month school year?

The following table summarizes the replies:

Number of years	Response to Question *b*			Total
	Yes	No	Don't know	
Less than 2	40	52	14	106
2–4	36	37	17	90
5–10	34	25	8	67
More than 10	46	41	3	90
Total	156	155	42	353

Let U = set of 353 public school teachers
A = set of teachers who answered "Yes"
B = set of teachers who answered "No"
C = set of teachers who have taught less than 2 years
D = set of teachers who have taught 2 to 4 years
E = set of teachers who have taught 5 to 10 years

a. Find the number of teachers in each of the following sets:

(1) $A \cup C$ (2) $A \cap D$
(3) $(A \cup B)' \cap E$ (4) $(A \cap C)'$
(5) $(A \cup B) \cap C$ (6) $(C \cup D) \cap (A \cup B)'$

b. Represent each of the following sets by using only the symbols A, B, C, D, E, $'$, $\cap$, and $\cup$:

(1) The set of teachers who have taught less than 5 years and who answered "Yes" *Answer:* $(C \cup D) \cap A$
(2) The set of teachers who have taught less than 10 years and who answered "No"
(3) The set of teachers who have taught more than 10 years and who answered "Yes"
(4) The set of teachers who answered "Don't know"
(5) The set of teachers who have taught more than 10 years and who answered "don't know"

13. The following exercise pertains to plane geometric figures. Which of the statements are true and which are false?

Let A = set of quadrilaterals
B = set of squares
C = set of rhombuses
D = set of triangles
E = set of parallelograms
F = set of rectangles
G = set of equilateral triangles
H = set of isosceles triangles
I = set of trapezoids
J = set of scalene triangles
K = set of right triangles

a. $B \subseteq I$ *b.* $B \subset C \subset E \subset A$ *c.* $G \cup H \cup J = D$
d. $C \cap F = B$ *e.* $I \cap E = \emptyset$ *f.* $H \cap G = H$
g. $H \cap G = G$ *h.* $G \cap K = J$ *i.* $K \cap G = \emptyset$
j. $K \subset J$ *k.* $(H \cup G)' \cap D = J$ *l.* $I \subseteq E$
m. $G \neq H$ *n.* $K \cap H = \emptyset$

14. In Section 1.5 it was shown that the set of natural numbers was equivalent to the set of even natural numbers. Not only is $E \leftrightarrow N$ but $E \subset N$ (E is a proper subset of N). Richard Dedekind (1831–1916) used this property to define an infinite set.

A set W is an infinite set if there exists a proper subset of W that is equivalent to W.

a. Show that the set $B = \{1,4,9,16,25, \ldots\}$ is equivalent to the set

$$N = \{1,2,3,4,5,6, \ldots\}$$

Is $B \subset N$?

b. Show that the set $D = \{5,10,15,20, \ldots\}$ is equivalent to the set

$$N = \{1,2,3,4,5, \ldots\}$$

Is $D \subset N$?

c. Show that the set $T = \{10,20,30,40, \ldots\}$ is an infinite set by placing it in 1-1 correspondence with one of its proper subsets.

d. Formulate a definition for a finite set in terms of a proper subset.

2

Real Numbers and Conditions

2.1 INTRODUCTION

Such concepts as "equations and inequalities," "absolute value," and "properties of a number system" are of major concern in modern mathematics and may be discussed through the language of sets. Consequently, the basic objectives of this chapter are:

a. To investigate the number systems familiar to us from arithmetic and algebra, noting in particular that the real-number system is an extension of various other number systems

b. To examine the use of conditions when expressed as equations or inequalities in the description of specific sets

c. To reemphasize and extend certain important concepts of Chapter 1 in the environment of the real-number system

d. To illustrate procedure and format for proofs of theorems which result as logical consequences of the postulates of the real-number system

2.2 CONCEPT OF A NUMBER SYSTEM

Familiarity with our surroundings is largely a matter of degree. Special characteristics and features possessed by people or objects often go by unnoticed. For example, we work, study, and spend countless hours in various buildings without a complete realization of their many distinct properties. Each building has been created according to a plan where specific materials were used for its construction and where the design was evolved to meet certain utilitarian and artistic objectives. The characteristics possessed by different types of structures tend to classify them as school buildings, homes, office buildings, and so on. Regardless of this classification, activities are carried on within them without any special concern for their distinct features. We become so accustomed to a particular structure that many of its subtle features are taken for granted.

In a similar manner we use numbers in everyday activities without observing too closely what is really known about them. When calculations are performed on these numbers, advantage is taken of the specific properties possessed by them. Everyone seems to be aware of the property that the sum of two integers is an integer or that the property $4 \cdot 3$ yields the same result as $3 \cdot 4$. In order to have a better understanding of such properties it is necessary to examine more closely the set of natural numbers, the set of integers, the set of rational numbers, and the set of real numbers. Each of these sets of numbers, together with the operations of addition and multiplication and certain assigned properties, forms a "number system." A number system consists of a set of objects (elements) combinable under two binary operations, called addition and multiplication, where:

a. Addition and multiplication are closed.
b. Addition and multiplication are commutative.
c. Addition and multiplication are associative.
d. Multiplication is distributive with respect to addition.

It is important to note that the binary operations (those involving two elements) need not be defined as addition and multiplication familiar to us from arithmetic and, further, that the objects need not be restricted to numbers. Such situations will be examined in a later chapter. For the present, the interpretation of objects and operations will be made in terms of those familiar number systems from arithmetic and algebra, extensions of which finally lead to the real-number system.

2.3 NATURAL NUMBERS

We recall that our earliest experiences with numbers involved those used specifically for counting, namely, the natural numbers or the positive integers, 1, 2, 3, 4, ... Progressively, we learned to perform the operations of addition, subtraction, multiplication, and division. If attention is now centered upon the two binary operations of addition and multiplication with respect to natural numbers, the following laws or postulates hold.

If $a, b, c, \ldots$ are elements of the set of natural numbers, N, combinable under the binary operations of addition, $(+)$ and multiplication $(\cdot)$, then:

Closure Laws

N-1: If $a \in N$ and $b \in N$, then $a + b \in N$.
N-2: If $a \in N$ and $b \in N$, then $ab \in N$.

Commutative Laws

N-3: If $a \in N$ and $b \in N$, then $a + b = b + a$.

N-4: If $a \in N$ and $b \in N$, then $ab = ba$ (ab or $a \cdot b$ means a multiplied by b).

Associative Laws

N-5: If $a \in N$, $b \in N$, and $c \in N$, then $(a + b) + c = a + (b + c)$.

N-6: If $a \in N$, $b \in N$, and $c \in N$, then $(ab)c = a(bc)$.

Distributive Law

N-7: If $a \in N$, $b \in N$, and $c \in N$, then $a(b + c) = ab + ac$ or $(b + c)a = ba + ca$ from N-4.

Identity Law

N-8: There exists in N a unique element "unity," designated as 1 and called the identity element for multiplication, such that $a \cdot 1 = 1 \cdot a = a$.

Illustrating these laws, we have:

N-1 and N-2: Since $3 \in N$ and $5 \in N$, then $3 + 5$ or $8 \in N$ and $3 \cdot 5$ or $15 \in N$.

N-3 and N-4: Since $6 \in N$ and $7 \in N$, then $6 + 7 = 7 + 6$ and $6 \cdot 7 = 7 \cdot 6$.

N-5 and N-6: Since $6 \in N$, $5 \in N$, and $3 \in N$, then $(6 + 5) + 3 = 6 + (5 + 3)$ or $11 + 3 = 6 + 8$. Also $(6 \cdot 5) \cdot 3 = 6 \cdot (5 \cdot 3)$ or $30 \cdot 3 = 6 \cdot 15$.

N-7: Since $3 \in N$, $2 \in N$, and $5 \in N$, then $3 \cdot (2 + 5) = 3 \cdot 2 + 3 \cdot 5$ or $3 \cdot 7 = 6 + 15$.

N-8: Since $5 \in N$, then $5 \cdot 1 = 5$.

A graphical representation (though incomplete) of the set of natural numbers is shown in Fig. 18.

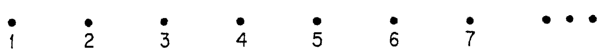


FIG. 18

The operations of subtraction and division are defined by means of addition and multiplication. Subtraction is defined through the equation $a - b = d$, which implies that if $a,b \in N$, then the subtraction of b from a is possible if there exists a natural number d such that $a = b + d$. We speak of d as the difference $a - b$. Similarly, division is defined

through the equation $a/b = q$, which implies that the division of a by b is possible if there exists a natural number q such that $a = bq$. Here q is called the quotient a/b, which is also written $a \div b$ or $a \cdot 1/b$.

For example, the subtraction $8 - 3 = d$ is possible if there exists some $d \in N$ such that $8 = 3 + d$. Similarly, the division $\frac{8}{2}$ is possible if there exists some q such that $8 = 2q$. In both cases, the operations can be performed, since d and q may be replaced by the natural numbers 5 and 4, respectively. As a consequence of the definition, subtraction is referred to as the inverse operation of addition, while division is referred to as the inverse operation of multiplication.

2.4 INTEGERS

Laws N-1 and N-2 state that addition and multiplication are always possible in the set of natural numbers; that is, if $a \in N$ and $b \in N$, then $a + b \in N$ and $ab \in N$. A similar situation exists only in a restricted sense for the operation of subtraction; or, in other words, the set of natural numbers is not closed in general with respect to subtraction. The testing of a few cases, such as $3 - 5$, $7 - 7$, and $4 - 9$, points out that no natural numbers exist which satisfy these differences. To remedy this situation the set of natural numbers is expanded so as to gain the property of closure with respect to subtraction. This requires the introduction of the negative integers and zero, which then, along with the positive integers, supply closure for all situations arising from $a - b = d$. The union of the set of natural numbers or positive integers and zero together with the set of negative integers is now the newly formed set of integers. We may write $N \subset I$ or $I^- \subset I$, where I^- represents the set of negative integers and I the set of integers. Laws N-1 through N-8, redesignated I-1 to I-8 with $a,b,c, \ldots \in I$, represent corresponding properties possessed by integers with respect to the operations of addition and multiplication. Hence, a number system called the "system of integers" has been formed.

The system of integers possesses the following additional properties not present in the system of natural numbers:

Identity Law

I-9: There exists in I a unique element zero (0), called the identity element for addition, such that $a + 0 = 0 + a = a$.

Inverse Element

I-10: For every a in I, there exists a unique element $(-a)$ of I, such that $a + (-a) = (-a) + a = 0$. Here $(-a)$ is called the additive inverse of a.

Closure Law for Subtraction

I-11: If $a \in I$ and $b \in I$, then $a - b \in I$.

As in the case with natural numbers, the operation of subtraction for integers is interpreted in terms of the operation of addition in the following manner:

Definition of Subtraction

I-12: If $a \in I$ and $b \in I$, then $a - b = a + (-b)$.

If familiarity with the rules of arithmetic (such as rules of signs) is assumed for operating with integers, the following examples illustrate properties I-1 through I-12.

Closure Laws: I-1 and I-2
Since $(-3) \in I$ and $(+5) \in I$, then $(-3) + (+5) \in I$ and $(-3)(+5) \in I$.
Commutative Laws: I-3 and I-4
Since $(-4) \in I$ and $(-2) \in I$, then $(-4) + (-2) = (-2) + (-4)$ and $(-4)(-2) = (-2)(-4)$.
Associative Laws: I-5 and I-6
Since $(-3) \in I$, $(-2) \in I$, and $(+5) \in I$, then $[(-3) + (-2)] + (+5) = (-3) + [(-2) + (+5)]$ and $[(-3)(-2)](+5) = (-3)[(-2)(+5)]$.
Distributive Law: I-7
Since $(-5) \in I$, $(+2) \in I$, and $(-4) \in I$, then $(-5)[(+2) + (-4)] = (-5)(+2) + (-5)(-4)$ or $(-5)(-2) = (10) + (+20)$.
Identity Laws: I-8 and I-9
Since $(-8) \in I$ and $(+7) \in I$, then $(-8)(1) = -8$ and $(+7) + 0 = +7$.
Inverse Element: I-10
Since $(-3) \in I$, then $(+3) \in I$ and $(-3) + (+3) = 0$.
Closure Law: I-11
Since $(-6) \in I$ and $(-5) \in I$, then $(-6) - (-5) = (-1)$ and $(-1) \in I$.
Definition of Subtraction: I-12
Since $(-9) \in I$ and $(+4) \in I$, then $(-9) - (+4) = -13$ and $(-9) + (-4) = -13$. Hence $(-9) - (+4) = (-9) + (-4)$.

A graphical representation of the set of integers (though incomplete) is shown in Fig. 19.

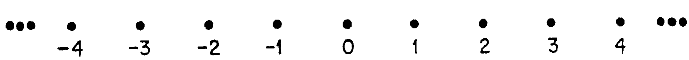


FIG. 19

2.5 RATIONAL NUMBERS

Even though the number system has thus far been expanded to include zero and negative integers, the operation of division is still not always possible. If $q = \frac{-8}{+3}$, then no integer q exists for which $q(+3) = -8$. A further extension of the system of integers is now necessary, but this must be accomplished without creating any inconsistencies that relate to Laws I-1 through I-12.

To accomplish this, we introduce the concept of a rational number, which is a number that can be expressed as the quotient p/q of two integers p and q, where q is not zero. Accordingly, the rational-number system is an extension of the system of integers, since every integer a may be written in the form $a/1$. For example, $3 = \frac{3}{1}$, $-5 = \frac{-5}{1}$, and $0 = \frac{0}{1}$. Thus, $I \subset F$ where F refers to the set of rational numbers.

The following rules from everyday arithmetic govern the operations with rational numbers:

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$

   Example: $\frac{3}{4} = \frac{6}{8}$ since $3 \cdot 8 = 4 \cdot 6$
2. $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

   Example: $\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$
3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

   Example: $\frac{3}{4} \cdot \frac{5}{7} = \frac{15}{28}$
4. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

   Example: $\frac{7}{8} \div \frac{2}{3} = \frac{21}{16}$
5. $\frac{ca}{cb} = \frac{a}{b}$

   Example: $\frac{5 \cdot 8}{5 \cdot 7} = \frac{8}{7}$

Each of these rules excludes zero denominators. It should be noted that a, b, and c may be thought of as being integers, but the rules apply equally well when $a,b,c \in F$.

For rational numbers the operation of division indicated by

$$a \div b = \frac{a}{b} = a\left(\frac{1}{b}\right)$$

where $a,b \in F$ and $b \neq 0$, is defined in terms of the operation of multiplication. For the case when $b = 0$, let $q = a/0$ where $a \neq 0$. In this instance there exists no rational-number replacement for q which yields a true statement, since $q \cdot 0 = 0$ for all $q \in F$. For $a = 0$ it follows that $q = \frac{0}{0}$ and $q \cdot 0 = 0$ is true for all $q \in F$. In the first case no rational number exists for q, while in the second no unique rational number exists for q. In all other cases of division, such as $q = \frac{12}{3}$, q is replaceable by a unique rational number. Hence $a/0$ (where $a \neq 0$) and $\frac{0}{0}$ are excluded from consideration, since no precise meaning can be attached to them.

It can now be shown that rational numbers satisfy Laws I-1 through I-12 of the system of integers. If, in I-1 through I-2, I is replaced everywhere by F and the term "integer" by "rational number," F-1 through F-12 are obtained, which, together with Laws F-13 and F-14, form the laws of the system of rational numbers.

F-13: For every $a \in F$ and $a \neq 0$, there exists an element $1/a$ such that $a(1/a) = (1/a)a = 1$. $1/a$ is called the multiplicative inverse or reciprocal of a and may be written a^{-1}.
Example: Since $3 \in F$, there exists an element of F, namely, $\frac{1}{3}$, such that $3(\frac{1}{3}) = 1$. Similarly, since $(-\frac{2}{3}) \in F$, then $-\frac{3}{2}$ (which also belongs to F) satisfies the requirement that $(-\frac{2}{3})(-\frac{3}{2}) = 1$.

F-14: If $a \in F$ and $b \in F$ and $b \neq 0$, then $a/b \in F$.
Example: Since $2 \in F$ and $5 \in F$, then $\frac{2}{5} \in F$. Similarly, if $\frac{3}{2} \in F$ and $\frac{4}{3} \in F$, then $\frac{3}{2}/\frac{4}{3} = \frac{9}{8}$, which is an element in F.

2.6 REAL NUMBERS AND THE COORDINATE LINE

Already two extensions of number systems serving special purposes have been made. The first extension of natural numbers to integers made subtraction always possible, while the second extension from integers to rational numbers made division always possible (zero excluded). The results obtained by these extensions are viewed in another way by saying that a solution to the equation $x + a = b$, where $a,b \in I$, always exists in the system of integers and that the solutions to both $ax = b$ (with $a \neq 0$) and $x + a = b$, where $a,b \in F$, always exist in the system of rational numbers. These equations or statements are referred to as first-degree equations in the one variable x of the form $ax + b = 0$ (with $a \neq 0$), where $a,b \in F$. In this case the solution set is $\{-b/a\}$ with the rational number $-b/a$ as the single element. Since the only operations involved in this type of first-degree equation in x are those of addition and multiplication, we refer to it as an algebraic equation.

Algebraic equations of the second degree in one variable x, with

coefficients restricted to the set of rational numbers, are typified by the form $ax^2 + bx + c = 0$ (with $a \neq 0$) and may or may not have solution sets containing rational numbers as elements. For example, the elements of the solution set $\{\frac{2}{3}, -\frac{2}{3}\}$ for the equation $x^2 - \frac{4}{9} = 0$ are rational numbers. However, the elements of the solution sets for the equations $x^2 - 5 = 0$ and $x^2 - 2 = 0$ are not rational numbers. In order to determine the respective solution sets for $x^2 - 2 = 0$ and $x^2 - 5 = 0$, a further extension of the system of rational numbers is made. Numbers of the form $\sqrt{2}$, π, log 5, and many others are not rational numbers since they cannot be expressed as exact quotients p/q of two integers p and q $(q \neq 0)$. These numbers are called irrational numbers and provide the source which assures us that every positive rational number will have a square root, a cube root, a logarithm, a sine, a cosine, and so on. Irrational numbers can be approximated through the use of rational numbers. For example, the rational number 3.1416 or $\frac{31416}{10000}$ has long been used as an approximation for π; similarly, 1.4142 has been used for $\sqrt{2}$. Various tables, such as those for logarithms, square roots, and cube roots, are other instances where rational numbers are used as approximations for irrational numbers. It should be noted that not all entries in tables require rational approximations; for example, log 1000 and sin 30° are rational numbers, namely, 3 and $\frac{1}{2}$, respectively. If the union of the set of rational numbers and the set of irrational numbers is formed, the set of all real numbers, designated as R_e, is obtained.

Real numbers may be represented geometrically as points on a straight line where a 1-1 correspondence is established between these points and the real numbers. A point is selected on this line and designated as its "zero point." Another point is then selected and designated as "one." With a scale now established and a point associated with zero, negative real numbers are associated with points to the left of zero and positive real numbers with points to the right of zero. The real number associated with each point is called its coordinate; and the line which now represents a geometrical picture of R_e is referred to as a coordinate line, a coordinate axis, or the real-number line.

No difficulty is experienced when associating points on the coordinate line with either integers or rational numbers. For example, the point associated with $-\frac{5}{4}$ requires that the unit between -1 and -2 be subdivided into four equal parts, and the first subdivision point to the left of -1 is associated with $-\frac{5}{4}$. The rational numbers themselves will not exhaust all the points on the real-number line. The points which are still unaccounted for correspond to those real numbers that are irrational. To illustrate what is meant by such a point, the real number $\sqrt{2}$ is located by constructing an isosceles right triangle on the coordinate line where one of its two equal legs coincides with the segment from 0 to 1, and the vertex of the right angle coincides with the point corresponding

to 1 as shown in Fig. 20. By the pythagorean theorem, $a^2 = 1^2 + 1^2 = 2$ and $a = \sqrt{2}$. If a circle is now drawn using the line segment a as its radius, it will intersect the coordinate line at the point A, which is then associated with the irrational number $\sqrt{2}$.

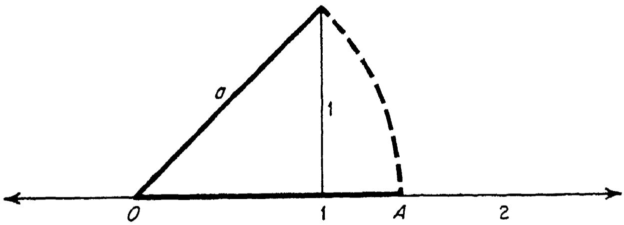


FIG. 20

A partial geometric picture of the set of real numbers or "the real-number line" is shown in Fig. 21. This line exhibits the different types of real numbers used in elementary mathematics.

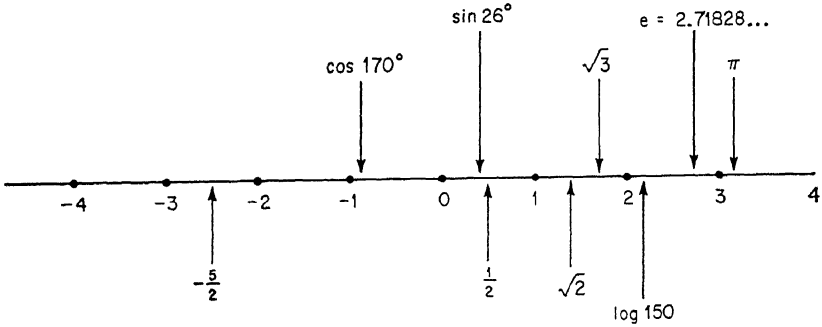


FIG. 21

Real numbers may be represented in terms of infinite (nonterminating) decimals. Integers are described in such a form through the use of zeros; for example, $6 = 6.000000 \cdots$ and $17 = 17.0000 \cdots$. In fact, rational numbers, which include integers, are expressible in decimal form by carrying out the indicated division between numerator and denominator. These decimal equivalents are of two types: either they will supposedly terminate, such as $\frac{7}{8} = 0.875 = 0.8750000 \cdots$; or they will continue forever, repeating a block of digits such as

$$\tfrac{3}{7} = 0.428571428571 \cdots$$

If the decimal equivalent is one of the type that terminates, which means it has a string of zeros at the right, the corresponding real number is definitely a rational number. If the decimal equivalent is nonterminating

but possesses the property of repeating a block of digits, such as

$$\tfrac{3}{7} = 0.428571428571 \cdots$$

then the corresponding real number is again a rational number. Irrational numbers may also be represented as infinite decimals, where no block of digits ever repeats, such as $\sqrt{2} = 1.4142136 \cdots$. In summary, all real numbers can be represented by infinite decimals.

Example 1. The determination of the equivalent rational form p/q for the corresponding decimal representation $0.232323 \cdots$ is obtained as follows.

Let $p/q = 0.232323 \cdots$ and multiply through by 10^n, where n represents the number of digits in the block that is repetitive. Since in $0.232323 \cdots$ a block of two digits is repetitive, we multiply by 10^2. Thus, $100\,p/q = 23.2323 \cdots$ and $p/q = 0.232323 \cdots$.

Subtracting p/q from $100\,p/q$ and, correspondingly, $0.2323 \cdots$ from $23.2323 \cdots$, we obtain $99\,p/q = 23$, from which $p/q = \tfrac{23}{99}$.

Hence, $\tfrac{23}{99} = 0.232323 \cdots$.

2.7 PROPERTIES OF THE REAL-NUMBER SYSTEM

Table 1 summarizes the properties of the different number systems discussed in this chapter. "Yes" indicates that the number system satisfies the property, while "No" indicates the opposite. The letter S acts as a placeholder for N, I, F, and R_e, and $a,b,c, \ldots \in S$.

Table 1

Property	S			
	N	I	F	R_e
1. $a + b \in S$	Yes	Yes	Yes	Yes
2. $ab \in S$	Yes	Yes	Yes	Yes
3. $a + b = b + a$	Yes	Yes	Yes	Yes
4. $ab = ba$	Yes	Yes	Yes	Yes
5. $(a + b) + c = a + (b + c)$	Yes	Yes	Yes	Yes
6. $(ab)c = a(bc)$	Yes	Yes	Yes	Yes
7. $a(b + c) = ab + ac$	Yes	Yes	Yes	Yes
8. $1 \in S$	Yes	Yes	Yes	Yes
9. $0 \in S$	No	Yes	Yes	Yes
10. $a - b \in S$	No	Yes	Yes	Yes
11. $\frac{a}{b} \in S$ with $b \neq 0$	No	No	Yes	Yes
12. $(-a) \in S$	No	Yes	Yes	Yes
13. $\frac{1}{a} \in S$ with $a \neq 0$	No	No	Yes	Yes

Any mathematical structure satisfying all these properties is called a field. Thus we have the "field of rational numbers" and the "field of real numbers," since all the laws for a field are satisfied in these two systems. In Chapter 5 the concept of a field is discussed in more detail. An intuitive acceptance of these properties for a specific S may be realized by replacing a, b, and c with the numbers of arithmetic in statements 1 to 13 and checking the results obtained.

The laws or postulates of a number system are usually expressed in the form of sentences where many different mathematical symbols are employed. Frequently, these sentences appear as equations or inequalities which communicate concisely and completely the properties of a mathematical system. They enable the mathematical system to exhibit a certain form or structure that distinguishes it from other systems.

A basic objective in this chapter is to expose various properties of the "real-number system" without entering into any detailed presentation of the development of the real-number system. The aim is to become aware of significant techniques and to justify some of the basic rules of algebra and arithmetic. To accomplish this more effectively, the laws of the real-number system are repeated for emphasis.

If $a, b, c, \ldots$ are real numbers that are combinable under the binary operations of addition $(+)$ and multiplication $(\cdot)$, then:

Closure Laws

R_e-1: If $a \in R_e$ and $b \in R_e$, then $a + b \in R_e$.

R_e-2: If $a \in R_e$ and $b \in R_e$, then $ab \in R_e$.

Commutative Laws

R_e-3: If $a \in R_e$ and $b \in R_e$, then $a + b = b + a$.

R_e-4: If $a \in R_e$ and $b \in R_e$, then $ab = ba$.

Associative Laws

R_e-5: If $a \in R_e$, $b \in R_e$, and $c \in R_e$, then $(a + b) + c = a + (b + c)$.

R_e-6: If $a \in R_e$, $b \in R_e$, and $c \in R_e$, then $(ab)c = a(bc)$.

Distributive Law

R_e-7: If $a \in R_e$, $b \in R_e$, and $c \in R_e$, then $a(b + c) = ab + ac$.

Identity Laws

R_e-8: There exists in R_e a unique element unity (1), called the identity element for multiplication, such that $a \cdot 1 = 1 \cdot a = a$.

R_e-9: There exists in R_e a unique element zero (0), called the identity element for addition, such that $a + 0 = 0 + a = a$.

Inverse Elements

R_e-10: For every a in R_e, there exists a unique element $(-a)$ of R_e, such that $a + (-a) = (-a) + a = 0$.

R_e-11: For every a in R_e, except 0, there exists a unique element $1/a$ of R_e, such that $a(1/a) = (1/a)a = 1$.

Definitions of Inverse Operations

Subtraction: $a - b = a + (-b)$

Division: $a \div b = a/b = a(1/b)$ where $b \neq 0$

When equality is expressed between real numbers, it is employed as an equivalence relation possessing the properties of reflexivity, symmetry, and transitivity. These properties and those essential to the operations of addition and multiplication follow.

Equality Properties

E-1, reflexive property: For $a \in R_e$, then $a = a$.

E-2, symmetric property: If $a \in R_e$, $b \in R_e$, and $a = b$, then $b = a$.

E-3, transitive property: If $a \in R_e$, $b \in R_e$, $c \in R_e$, $a = b$, and $b = c$, then $a = c$.

E-4, addition property: If $a \in R_e$, $b \in R_e$, $c \in R_e$, and $a = b$, then $a + c = b + c$.

E-5, multiplication property: If $a \in R_e$, $b \in R_e$, $c \in R_e$, and $a = b$, then $ac = bc$.

Properties E-1 through E-3 will be discussed in more detail in Section 3.5.

A familiar rule in elementary algebra states, "If equals are added to equals, the sums are equal." The expression "equals" is a concise way of saying "equal numbers" and should be interpreted accordingly. In symbols, if $a = b$ and $c = d$, then $a + c = b + d$. This rule is now proved with the aid of the laws that govern the real-number system.

Example 1. If $a,b,c,d \in R_e$ and if $a = b$ and $c = d$, then $a + c = b + d$.
Proof:

Statement	*Authority*
1. $a = b$	1. Given
2. $a + c = b + c$	2. Step 1 and E-4
3. $c = d$	3. Given
4. $c + b = d + b$	4. Step 3 and E-4
5. $b + c = b + d$	5. R_e-3
6. $a + c = b + c$ and $b + c = b + d$	6. Steps 2 and 5
7. $a + c = b + d$	7. Step 6 and E-3

The proof of the rule which states "If equals are multiplied by equals, the products are equal" is obtained in an analogous manner.

Mathematical sentences of the types $ab = ba$, $a + c = c + a$, and $a(b + c) = ab + ac$ yield true statements whenever a, b, and c are replaced by real numbers. Such sentences or conditions are referred to as "identities" or "identical equations." Identities have already been encountered in sets such as $A = \{x \in R_e \mid x^2 - 1 = (x + 1)(x - 1)\}$. The condition $x^2 - 1 = (x + 1)(x - 1)$ is true of all real-number replacements for x. For example, if x is replaced by -3, then

$$x^2 - 1 = (-3)^2 - 1 = 8$$

while $(x + 1)(x - 1) = (-3 + 1)(-3 - 1) = (-2)(-4) = 8$.

Mathematical sentences of the types $2x - 3 = 5$ and $x^2 - 2x - 3 = 0$ yield valid statements only when x is replaced by a finite number of elements chosen from the set of real numbers. Thus, the solution set of $2x - 3 = 5$ is $\{4\}$; the solution set of $x^2 - 2x - 3 = 0$ is $\{3,-1\}$. Mathematical sentences of the types $x^2 - 2x - 3 = 0$ and $2x - 3 = 5$ are referred to as conditional equations. Mathematical statements of identity assert general truths about all numbers in a system of numbers and provide working rules which can be applied freely with any of the numbers in that system of numbers. Mathematical statements of condition impose restrictions on the variables involved and permit only the use of certain specified numbers in the system of numbers to which the statements apply.

Additional identities, usually called theorems, may be created from the basic list of laws for R_e. These theorems extend techniques of operation with real numbers so as to facilitate manipulative procedures. The following represent a few such theorems and are derivable from the original laws that govern real numbers.

If $a,b,c,d \in R_e$, then:

T-1: If $a = b$ and $c = d$, then $a + c = b + d$.
T-2: If $a = b$ and $c = d$, then $ac = bd$.
T-3: $ab = 0$ if and only if $a = 0$ or $b = 0$.
T-4: If $a + b = 0$, then $a = -b$.
T-5: If $a + b = c + b$, then $a = c$.
T-6: If $ab = cb$, then $a = c$ (where $b \neq 0$).
T-7: $-a = (-1)(a)$
T-8: $-(-a) = a$
T-9: $(-a)(-b) = ab$
T-10: $(-a)(b) = -ab$
T-11: $a(b - c) = ab - ac$
T-12: $(a^{-1})^{-1} = a$ (where $a \neq 0$)

T-13: $(a + b)(a + b) = a^2 + 2ab + b^2$
T-14: $(a - b)(a - b) = a^2 - 2ab + b^2$
T-15: $(a - b)(a + b) = a^2 - b^2$
T-16: $(a - b)(a^2 + ab + b^2) = a^3 - b^3$
T-17: $(a + b)(a^2 - ab + b^2) = a^3 + b^3$
T-18: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$
T-19: $(a + b)^3 = a^2 + 3a^2b + 3ab^2 + b^3$

The method of proof of certain selected theorems is illustrated in the following examples.

Example 2. Theorem 4: If $a + b = 0$, then $a = -b$.
Proof:

Statement	*Authority*
1. $a + b = 0$	1. Given
2. $(a + b) + (-b) = 0 + (-b)$	2. E-4
3. $a + [b + (-b)] = 0 + (-b)$	3. R_e-5
4. $a + 0 = 0 + (-b)$	4. R_e-10
5. $a = -b$	5. R_e-9

Example 3. Theorems 9 and 10 are customarily introduced in elementary algebra as "rules of signs." The proofs of these theorems follow:

Theorem 9: $(-a)(-b) = ab$

Proof:
Consider the two expressions $(-a)(b) + (-a)(-b)$ and $(-a)(b) + ab$.

	Authority
$(-a)(-b) + (-a)(b)$	
$= -a(-b + b)$	R_e-7
$= -a(0)$	R_e-10
$= 0$	T-3
$(-a)(b) + ab = b(-a + a)$	R_e-7
$= b(0)$	R_e-10
$= 0$	T-3

Since $(-a)(-b) + (-a)(b) = 0$ and $0 = (-a)(b) + ab$, then by E-3, $(-a)(-b) + (-a)(b) = (-a)(b) + ab$. Hence by R_e-3,

$$(-a)(-b) + (-a)(b) = ab + (-a)(b)$$

and by T-5, $(-a)(-b) = ab$.

Theorem 10: $(-a)(b) = -ab$

Proof:

Consider the expression $(-a)(b) + ab$.

		Authority
$(-a)(b) + ab$	$= (-a + a)b$	R_e-7
	$= (0)b$	R_e-10
	$= 0$	T-3

Theorem 4 states that if $a + b = 0$, then $a = -b$. Since

$$(-a)(b) + ab = 0$$

then by T-4, $(-a)(b) = -ab$.

2.8 FACTORIZATION OF POLYNOMIALS

The list of theorems in the previous section includes many identities that are referred to as "type products." These were used in the factorization of polynomials in algebra and provided a means of recognizing certain basic patterns for purposes of factoring. The identity

$$a^2 - b^2 = (a + b)(a - b)$$

is frequently referred to as the "difference of two squares"; similarly, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ as the "sum of two cubes";

$$(a + b)^2 = a^2 + 2ab + b^2$$

as the "perfect square trinomial"; and many others.

The proof of the identity $a^2 - b^2 = (a + b)(a - b)$ appears below.

Proof:	*Authority*
$(a - b)(a + b)$	
$= (a - b)a + (a - b)b$	R_e-7
$= a(a - b) + b(a - b)$	R_e-4
$= a^2 - ab + ba - b^2$	R_e-7
$= a^2 - ab + ab - b^2$	R_e-4
$= a^2 + (-ab + ab) - b^2$	R_e-5
$= a^2 - b^2$	R_e-10, R_e-5, R_e-9

It is customary to speak of a "polynomial in the variable x" as an expression of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$, where the exponents are nonnegative integers and the coefficients $a_n, a_{n-1}, \ldots$ are specific elements chosen from some designated set such as R_e, F, or I. According to this definition, $3x^4 - 2x^3 - 4x$, $\frac{1}{2}x^3 - 3x$, $2x - 1$, and 5 are

polynomials in the variable x of degree 4, 3, 1, and 0, respectively, with coefficients chosen from R_e or one of its subsets. The degree of the polynomial is determined by the largest exponent of x, namely, n, if $a_n \neq 0$. Nonzero constants such as 5 and 6 are referred to as constant polynomials of zero degree, since $5 = 5x^0$. The constant zero itself is called the zero polynomial and has no degree associated with it. We shall be concerned primarily with nonconstant polynomials.

It is also customary to speak of a polynomial in the variables $x, y, z, \ldots$ as either a single term or the algebraic sum of terms of the form $bx^m y^n z^p \cdots$, where b is called the coefficient of the term and is a specific element chosen from some designated set such as $R_e, F, I, \ldots$ and where each of the exponents $m, n, p, \ldots$ is either a positive integer or zero. Expressions of the type $5x^2 - 2xyz + 3$, -7, $2x + y$, x, $3x^3y - 2z^5w - 1$, $-4x^3y^2z, \ldots$ fulfill the requirements for being such a polynomial.

The factorization of a polynomial is dependent upon the universe from which the coefficients involved in the factors are to be chosen. For example, $x^2 - 4$ is factorable into $(x + 2)(x - 2)$ if the coefficients appearing in the factors are chosen from the set of integers. If, however, the coefficients are restricted to the set of natural numbers, then $x^2 - 4$ is not reducible. If the factorization of a given polynomial requires the introduction of new polynomials whose coefficients are not elements of the designated universe, then we say that the original polynomial is "not reducible." In this instance the original polynomial is considered as its best factored form. The polynomial $x^2 + 4$ is not reducible if the coefficients are restricted to the set of real numbers, but

$$x^2 + 4 = (x + 2i)(x - 2i)$$

if the coefficients are permitted to be elements in the set of complex numbers. When only natural numbers or integers are available for the coefficients, then the polynomial $x^2 + x + \frac{1}{4}$ is not reducible, but it is readily seen to be $(x + \frac{1}{2})^2$ when rational numbers are permitted. Table 1, where n.r. means "not reducible," offers additional examples.

Table 1

Polynomial	N	I	F	R_e
a. $x^2 - 9$	n.r.	$(x+3)(x-3)$	$(x+3)(x-3)$	$(x+3)(x-3)$
b. $4x^2 - 4x + 1$	n.r.	$(2x-1)^2$	$(2x-1)^2$	$(2x-1)^2$
c. $16y^2 - \frac{1}{9}$	n.r.	n.r.	$(4y+\frac{1}{3})(4y-\frac{1}{3})$	$(4y+\frac{1}{3})(4y-\frac{1}{3})$
d. $x^3 + 27$	n.r.	$(x+3)(x^2-3x+9)$	$(x+3)(x^2-3x+9)$	$(x+3)(x^2-3x+9)$
e. $\frac{1}{2}x^2 - \frac{1}{4}x$	n.r.	n.r.	$\frac{1}{2}x(x-\frac{1}{2})$	$\frac{1}{2}x(x-\frac{1}{2})$
f. $x^2 - 3$	n.r.	n.r.	n.r.	$(x+\sqrt{3})(x-\sqrt{3})$
g. $x^2 + 8$	n.r.	n.r.	n.r.	n.r.

It should be noted that a polynomial may be factored in more than one way and still satisfy the requirements relative to the choice of coefficients. In row e of Table 1, other factored forms could also have been stated such as $\frac{1}{4}x(2x - 1)$, which would have been more serviceable in a problem like $(\frac{1}{2}x^2 - \frac{1}{4}x)/(2x - 1)$ for simplification purposes. The context of a problem will usually indicate the direction and type of factorization most desirable. If the exponents are permitted to be chosen from the set of rational numbers and if it is stipulated that all coefficients be real numbers, the algebraic expression $x - 4$ has factored forms such as $(x^{\frac{1}{2}} - 2)(x^{\frac{1}{2}} + 2)$ or $(x^{\frac{1}{3}} - 4^{\frac{1}{3}})(x^{\frac{2}{3}} + 4^{\frac{1}{3}}x^{\frac{1}{3}} + 4^{\frac{2}{3}})$ and many others.

2.9 SOLUTION SETS

In finding the solution set of a first-degree or linear equation in one variable, $ax + b = 0$ $(a \neq 0)$, the laws and theorems governing R_e when applied result in a condition or equation expressible as $x = c$. These simpler conditions of the form $x = c$ determine the solution set of the original equation, namely, $\{c\}$, or $\{-b/a\}$.

Example 1. The solution set of $3x - 5 = 1$, where $x \in R_e$, is obtained as follows:

	Authority
$3x - 5 = 1$	Given
$3x - 5 + 5 = 1 + 5$	E-4
$3x = 6$	R_e-10
$\frac{1}{3}(3x) = \frac{1}{3}(6)$	E-5
$x = 2$	R_e-11

The solution set of $3x - 5 = 1$ is $\{2\}$.

The original equation $3x - 5 = 1$ yields the same solution set as $x = 2$; that is, $\{x \in R_e \mid 3x - 5 = 1\} = \{x \in R_e \mid x = 2\} = \{2\}$. Equations resulting in the same solution set are called equivalent equations.

The solution sets of second-degree equations or quadratic equations in one variable are obtained in a similar manner. From the quadratic equation $ax^2 + bx + c = 0$ (with $a \neq 0$, $a,b,c \in R_e$), two first-degree equations in one variable can be obtained from which the desired solution set is readily determined. However, it is noted that if the variable $x \in R_e$, the solution set of the quadratic equation in many instances may be the null set. For example, the solution set of the quadratic equation $x^2 - 2 = 0$ is $\{\sqrt{2}, -\sqrt{2}\}$ if $x \in R_e$. However, the solution set of $x^2 + 1 = 0$ is the null set if $x \in R_e$ but is $\{i, -i\}$ if the universe of the variable x is the set of complex numbers. The solution set of a

quadratic equation $ax^2 + bx + c = 0$ depends upon Theorem 3 (Section 2.7), which states that a product ab is zero if and only if at least one of its factors, a or b, is zero.

Example 2. The solution set of $3x^2 - 4x + 1 = 0$, where $x \in R_e$, may be determined as follows:

$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$

$$\begin{array}{ccc} 3x - 1 = 0 & \qquad & x - 1 = 0 \\ | & & | \\ x = \frac{1}{3} & & x = 1 \end{array}$$

If x is replaced by $\frac{1}{3}$, then $3x - 1 = 0$, $x - 1 = -\frac{2}{3}$, and $(0)(-\frac{2}{3}) = 0$. Further, if x is replaced by 1, then $3x - 1 = 2$, $x - 1 = 0$, and $2 \cdot 0 = 0$. Consequently, the solution set of $3x^2 - 4x + 1 = 0$, where $x \in R_e$, is $\{\frac{1}{3},1\}$. It follows that

$$\{x \in R_e \mid 3x^2 - 4x + 1 = 0\} = \{x \in R_e \mid x = \tfrac{1}{3} \text{ or } x = 1\} = \{\tfrac{1}{3},1\}$$

Exercise 10

1*a*. Which of the following are rational numbers? Irrational numbers? Integers? $\sqrt{4}$, $-\sqrt{3}$, $-\sqrt{\frac{4}{9}}$, $\frac{1}{10}$, $\sqrt{12}$, $-\sqrt{16}$, $\sqrt{9}$, $\sqrt{\frac{9}{16}}$, $2\sqrt{5} - 1$, $-3\sqrt{4}$, π, $\dfrac{\sqrt{16}}{\sqrt{4}}$, $\frac{8}{2}$, 3.14159, 0.2171828.

b. Which of the following are real numbers? $\sqrt{9}$, $\sqrt{-9}$, $3\sqrt{4}$, $-3\sqrt{4}$, $-\sqrt{9}$, $-\sqrt{-9}$, $\dfrac{\sqrt{3}}{\sqrt{2}}$, $\sqrt{2} + 2$, $\sqrt{3} - \sqrt{2}$, $\sqrt{-3} - \sqrt{2}$, $\sqrt[3]{-27}$, $-\sqrt[3]{27}$, $\sqrt[3]{-\frac{64}{125}}$, $\sqrt{-\frac{64}{125}}$.

c. If $x \in N$, which of the following expressions are always natural numbers? $x + 2$, $x - 1$, $x^2 - 4$, x^2, $-x$, $3 - x$, $2x$, $\dfrac{x}{3}$, $x + 2x$, $2x^2 - x$, $x^2 - x\sqrt{x}$.

d. If $x \in I$, determine which of the expressions in part *c* always represent integers.

2. Determine which of the following statements are true:

a. $8 \div 5 \in I$	*b*. $0 \div 5 \in R_e$	*c*. $\frac{0}{3} \in N$
d. $\frac{3}{0} \in F$	*e*. $\frac{0}{3} \in I$	*f*. $4 - 4 \in I$
g. $4 - 4 \in N$	*h*. $\frac{0}{0} \in R_e$	*i*. $\frac{18}{2} \in I$
j. $7 \cdot 0 \in I$	*k*. $\frac{17}{3} \in F$	*l*. $4 - 7 \in N$
m. $7 - 4 \in N$		

3. Using real numbers, show by a suitable counterexample that the following statements are true:

a. Subtraction is not commutative.

b. Subtraction is not associative.
c. Division is not commutative.
d. Division is not associative.
e. Addition will not distribute over multiplication.

4. Determine the additive inverse and the multiplicative inverse for each of the following (it is understood that $x \in R_e$):

a. 2	*b.* -3	*c.* $\frac{1}{2}$
d. $-\frac{2}{5}$	*e.* $\frac{3}{4}$	*f.* $-\frac{7}{5}$
g. 0	*h.* 1	*i.* $-3\frac{1}{4}$
j. $\sqrt{5}$	*k.* $-\sqrt{3}$	*l.* $-\pi$
m. $x - 3$	*n.* $3x$	*o.* $\frac{2}{x} - 1,\ x \neq 0$
p. $-\frac{x-2}{x+2},\ x \neq -2$		

5. Determine whether closure exists for each of the specified sets with respect to the indicated operations.

Set	Addition	Subtraction	Multiplication	Division
Positive integers				
Negative integers				
Positive rational numbers				
Negative real numbers				
Primes				
Positive even integers				
Positive odd integers				

6. Factor each of the following polynomials. Indicate clearly the laws or theorems used as your authority. Assume that the coefficients are elements of R_e.

Example:

$$\begin{aligned} 2x^2 - 3x - 5 &= 2x^2 + 2x - 5x - 5 && \text{Since } -3x = 2x - 5x \\ &= 2x(x+1) - 5(x+1) && R_e\text{-}7 \\ &= (x+1)(2x-5) && R_e\text{-}7 \end{aligned}$$

a. $4 - a^2$	*b.* $8b^3 - 1$
c. $x^4 - x$	*d.* $c^4 + 10c^2 + 16$
e. $6 - y - y^2$	*f.* $x^2 + 3x + 5$
g. $9x^2 - 30x + 25$	*h.* $1 - 16a^2x^2$
i. $50a^2 - 32b^2$	*j.* $27x^3 + 8y^3$
k. $64x^6 - y^6$	*l.* $9x^2 - (y-2)^2$
m. $x^2 - 4x + 4 - (z+1)^2$	*n.* $(x - 4y)^2 - 3(x - 4y) - 10$
o. $4x + y - 12x^2 - 3xy$	*p.* $x^4 + 3x^2 + 4$
q. $3b^3 + 3b^2 - b - 1$	*r.* $x^2 + 2$

7. Complete the following table. If a polynomial is not reducible because of the type of coefficients specified, write n.r. If it is factorable, display the factors.

Polynomial	Type of coefficients			
	N	I	F	R_e
a. $x^2 + 3x$				
b. $x^2 - 16$				
c. $3x^2 - 9$				
d. $\frac{1}{3}x^3 - 3x^2$				
e. $\dfrac{3x}{a^2} + \dfrac{3x}{ab}$				
f. $5x^2 - 7$				
g. $3x^2 + 6x + 3$				
h. $3y^2 + 24y - 60$				
i. $x^2 + 4$				
j. $x^2 + \dfrac{2x}{a} + \dfrac{1}{a^2}$				
k. $x^3 - 16$				

8. Find the rational form p/q corresponding to each of the following:

a. $0.565656 \cdots$ *b.* $0.33333 \cdots$ *c.* $0.56785678 \cdots$
d. $2.363636 \cdots$ *e.* $1.37575 \cdots$ *f.* $0.3272727 \cdots$
g. $2.1636363 \cdots$ *h.* $0.693693 \cdots$

9. Since all real numbers may be represented as infinite decimals, represent each of the following in this manner:

a. 2 *b.* 0 *c.* $\dfrac{\sqrt{5}}{\sqrt{3}}$
d. $\frac{3}{4}$ *e.* 1 *f.* $\frac{31314}{10000}$
g. $\frac{2}{9}$ *h.* $3\frac{2}{5}$ *i.* $1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!}$
j. $\frac{3}{10}$ *k.* $6\frac{7}{23}$ *l.* $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}$
m. $\sqrt{3}$ *n.* $-3\frac{5}{19}$ *o.* $2 - \sqrt{3}$
p. $\frac{5}{11}$ *q.* $-3\sqrt{2}$ *r.* $3\sqrt{2} - 5\sqrt{3}$
s. $\frac{7}{13}$ *t.* $2 + \sqrt{3}$ *u.* $\dfrac{3\sqrt{2} - 5}{\sqrt{2} + 3}$

10. By use of laws R_e-1 through R_e-12 and theorems T-1 through T-10, prove each of the following identities, assuming that $a,b,c \in R_e$.

a. If $a = b$, then $ac = bc$. *b.* $(a + b)^2 = a^2 + 2ab + b^2$
c. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ *d.* $a(b - c) = ab - ac$

Using only laws R_e-1 through R_e-12, prove the following:

e. If $b + a = c + a$, then $b = c$.
f. If $ba = ca$, then $b = c$ (where $a \neq 0$).

11. Find the solution set for each of the following equations with respect to the designated universe:

Equation	$x \in N$	$x \in I$	$x \in F$	$x \in R_e$
a. $2x + 3 = 6x - 5$				
b. $7(x + 2) = 3x - 1$				
c. $\frac{2}{3} - \frac{3}{x} = 1$				
d. $\frac{x}{5} = \frac{2}{x}$				
e. $x^2 - 5x + 6 = 0$				
f. $x^2 - 3x - 18 = 0$				
g. $2x^2 - 5x + 3 = 0$				
h. $x^2 - 4x + 4 = 0$				
i. $x^2 - 3 = 0$				
j. $(x + 2)(x^2 - 2x + 4) = x^3 + 8$				
k. $x^4 + 9 = 0$				

2.10 CONCEPT OF ORDER

In Section 2.6 we agreed to associate points to the right of zero on the real-number line with positive numbers and those to the left of zero with negative numbers. To indicate that a real number r is positive, we write $r > 0$ and say "r is greater than zero," while if r is a negative real number, we write $r < 0$ and say "r is less than zero." Further, to introduce the concept of "less than" between pairs of real numbers, the following definition is made.

A real number "b" is less than a real number "a," written $b < a$, if there exists some positive real number y such that $a = b + y$.

The sentence $b < a$ implies the sentence $a > b$; that is, if b is less than a, then a is greater than b. We note that if $b < a$, the point on the real-number line associated with b lies to the left of the point associated with a. For example, since $3 + 2 = 5$, then $3 < 5$. On the real-number line the point associated with 3 lies to the left of the point associated with 5. Similarly, $-5 < 6$, since $11 + (-5) = 6$.

Sentences such as "$a \geq b$" and "$a \leq b$" are read "a is greater than or equal to b" and, correspondingly, "a is less than or equal to b." Symbolically $a \geq b$ implies $a > b \veebar a = b$ where the "exclusive or" ($\veebar$) indicates that one or the other is true but not both. Sentences employing the symbols $>$, $<$, $\leq$, or $\geq$ are referred to as inequalities.

Example 1. The symbols for inequality are frequently utilized in the writing of a condition or a defining property which then serves to define

a set. For each inequality a universe must be specified as the source from which elements are to be selected for testing in the defining condition. The following examples serve to illustrate these ideas.

a. If $A = \{x \in N \mid x < 4\}$, then $A = \{1,2,3\}$.

If $B = \{x \in N \mid x \leq 4\}$, then $B = \{1,2,3,4\}$.

Note that $4 \leq 4$ is a valid statement according to the meaning of "$\leq$."

b. If $A = \{x \mid x \text{ is a prime less than } 10\}$, then $A = \{2,3,5,7\}$. The elements of A are designated by black circles on the real-number line in Fig. 22.

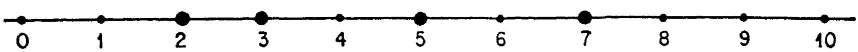


FIG. 22

c. Consider the set $A = \{x \in R_e \mid x < 3 \wedge x > -2\}$. Here each element of A must be less than 3 and at the same time greater than -2. This defining condition may also be written $-2 < x < 3$. In order to represent set A graphically, open circles are used at 2 and 3 to indicate their exclusion from the set of elements. The heavy line used to join these two circles represents the set of elements satisfying both conditions. Thus the graph of A is shown in Fig. 23.

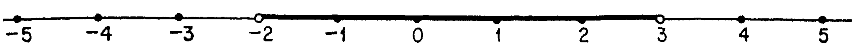


FIG. 23

d. Consider the set $B = \{x \in R_e \mid x \geq 3 \vee x < -2\}$. Each element in set B is equal to or greater than 3 or less than -2. (Notice that no elements here satisfy both conditions simultaneously.) For graphical purposes we use a black circle on the real-number line at 3 to indicate inclusion and an open circle at -2 to indicate exclusion. Thus, the graph of B is shown in Fig. 24.

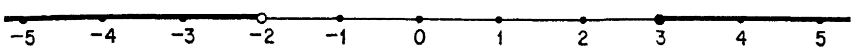


FIG. 24

An illustration of interest here is a set of the type

$$G = \{x \in R_e \mid x \geq -1 \vee x \leq 3\}$$

which should be compared with set B. Set G contains elements which satisfy the "either-or" condition as well as the "both" condition; i.e., there exist certain elements which are greater than -1 at the same time

they are less than 3. The set of elements satisfying each condition is illustrated by the diagrams shown in Fig. 25. It is noted that the set G

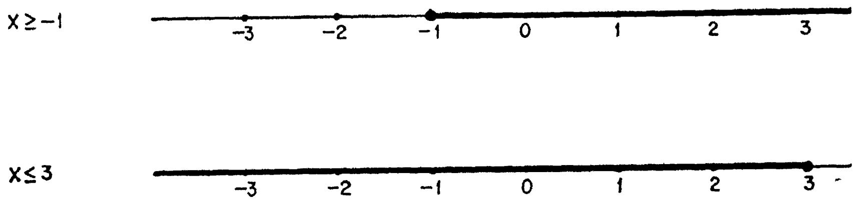


FIG. 25

can be replaced by the one set $\{x \in R_e\}$, which is graphically the entire real-number line, since G represents the union of the two sets of points.

2.11 ORDER PROPERTIES—SOLUTION OF INEQUALITIES

The manipulative procedures used in the solution of inequalities are very much the same as those used with equations. However, there are certain exceptions. To indicate some of the similarities and differences, the agreement is made that the inequality $3 > -2$ has the same "sense" as $6 > -4$, while $3 > -2$ has a "sense" opposite to that of $-6 < 4$. The sense of an inequality is preserved when the same real number is added to both members of an inequality or when both members are multiplied or divided by a positive real number. However, the sense of the inequality is reversed if both members of an inequality are multiplied or divided by a negative real number. These ideas are now used to summarize the order properties of real numbers.

If $a,b,c \in R_e$, then:

O-1: For any two real numbers a and b one and only one of the following is true:

$$a < b \qquad a = b \qquad a > b$$

O-2: If $a < b$ and $b < c$, then $a < c$.
O-3: If $a < b$, then $a + c < b + c$.
O-4: If $a < b$ and $c > 0$, then $ac < bc$.
O-5: If $a < b$ and $c < 0$, then $ac > bc$.

Example 1. We use the definition of $a < b$ to illustrate the proof of O-2: If $a < b$ and $b < c$, then $a < c$.

Proof: If $a < b$ and $b < c$, then by definition $a + x = b$ and $b + y = c$, where x and y are positive real numbers. By Theorem T-1 of Section 2.7,

$(a + x) + (b + y) = b + c$, from which it follows that $a + (x + y) = c$. Accordingly, $a < c$, since $(x + y)$ is a positive real number.

Example 2. Many theorems are logical consequences of the definition of order and the order properties. To illustrate, the proofs of two such theorems are included.

a. If $a < b$ and $c < d$, then $a + c < b + d$.

Proof: If $a < b$ and $c < d$, then $a + y = b$ and $c + z = d$, where y and z are positive real numbers. Hence

$$(a + y) + (c + z) = (a + c) + (y + z) = b + d$$

Consequently, since $(y + z)$ is a positive real number, $a + c < b + d$.

b. If $ab > 0$, then $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

Proof:

Case 1: Neither a nor b can be zero, since then $ab = 0$, which is contrary to our hypothesis.

Case 2: If either a or b is positive, say $a > 0$, and if $b < 0$, then $ab < 0$ (by O-5), which is contrary to hypothesis. Thus it follows that if either a or b is positive, the other must also be positive.

Case 3: If either a or b is negative, say $a < 0$, and if $b > 0$, then $ab < 0$ (by O-5), which is again contrary to hypothesis. Thus it follows that if either a or b is negative, then the other must also be negative.

The theorem "If $ab < 0$, then $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$" is left as an exercise.

Example 3. The following examples illustrate the use of the order properties as applied to the solution of inequalities.

a. To describe more concisely the defining condition of the set $\{x \in R_e \mid -3x - 2 > -2x - 4\}$, the aforementioned properties are applied to the condition $-3x - 2 > -2x - 4$ (a linear inequality).

$-3x - 2 > -2x - 4$	
$-3x > -2x - 2$	Adding 2 to both members
$-x > -2$	Adding $2x$ to both members
$x < 2$	Multiplying both members by (-1)

Hence

$$\{x \in R_e \mid -3x - 2 > -2x - 4\} = \{x \in R_e \mid x < 2\}$$

b. To state more concisely the defining condition of a set as $\{x \in R_e \mid x^2 - 3x < 4\}$, the quadratic inequality $x^2 - 3x < 4$ can be rewritten as $x^2 - 3x - 4 < 0$, which is factored $(x - 4)(x + 1) < 0$. This statement is true if one of the factors is negative and the other

positive. Thus

$$x^2 - 3x - 4 < 0$$

$$x - 4 > 0 \wedge x + 1 < 0 \quad \text{or} \quad x - 4 < 0 \wedge x + 1 > 0$$

$$x > 4 \wedge x < -1 \quad \text{or} \quad x < 4 \wedge x > -1$$

The compound condition $x > 4 \wedge x < -1$ has the null set as its solution set, since x cannot be greater than 4 and at the same time less than -1. The second compound condition has the set $\{x \in R_e \mid x < 4 \wedge x > -1\}$ as its solution set, since x can be less than 4 and at the same time greater than -1. It follows that

$$\{x \in R_e \mid x^2 - 3x < 4\} = \emptyset \cup \{x \in R_e \mid x < 4 \wedge x > -1\}$$

which yields the second set. Graphically, the solution set is represented as shown in Fig. 26.

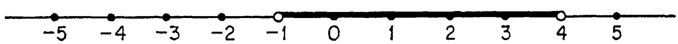


FIG. 26

c. To state more concisely the defining condition of a set as $\{x \in R_e \mid x^2 - 3x > 4\}$, the inequality can be rewritten $x^2 - 3x - 4 > 0$, which is factored $(x - 4)(x + 1) > 0$. Since the product of the two numbers $(x - 4)$ and $(x + 1)$ is greater than zero, either both factors are positive or both are negative. Thus

$$x^2 - 3x - 4 > 0$$

$$x - 4 > 0 \wedge x + 1 > 0 \quad \text{or} \quad x - 4 < 0 \wedge x + 1 < 0$$

$$x > 4 \wedge x > -1 \quad \text{or} \quad x < 4 \wedge x < -1$$

The compound conditions $x > 4 \wedge x > -1$ and $x < 4 \wedge x < -1$ are replaceable, respectively, by $x > 4$ and $x < -1$. Hence

$$\{x \in R_e \mid x^2 - 3x > 4\} = \{x \in R_e \mid x > 4 \vee x < -1\}$$

Graphically, the solution set is represented as shown in Fig. 27. Here

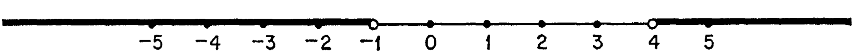


FIG. 27

-1 and 4 are excluded, because neither choice would satisfy $x^2 - 3x > 4$, since 4 is not greater than 4. It is noted that if

$$A = \{x \in R_e \mid x^2 - 3x - 4 > 0\}$$

$B = \{x \in R_e \mid x^2 - 3x - 4 = 0\}$, and $C = \{x \in R_e \mid x^2 - 3x - 4 < 0\}$, then $A \cup B \cup C = R_e$, which is represented graphically by the real-number line.

d. To state more concisely the defining condition of a set such as $\{x \in R_e \mid 2x^2 + 2x + 1 > 0\}$, we first observe that the equation $2x^2 + 2x + 1$ does not lend itself conveniently to factorization. Hence, a method other than factorization is employed for the study of $2x^2 + 2x + 1 > 0$. The method of completing squares is here used to advantage and illustrated as follows:

$$\begin{aligned}
&2x^2 + 2x + 1 > 0\\
&2x^2 + 2x > -1\\
&x^2 + x > -\tfrac{1}{2}\\
&x^2 + x + \tfrac{1}{4} > -\tfrac{1}{2} + \tfrac{1}{4}\\
&(x + \tfrac{1}{2})^2 > -\tfrac{1}{4}
\end{aligned}$$

The left member $(x + \frac{1}{2})^2$ is nonnegative for all real values of x. Hence, the inequality is true for all $x \in R_e$; i.e.,

$$\{x \in R_e \mid 2x^2 + 2x + 1 > 0\} = \{x \mid x \in R_e\}$$

It is interesting to note that the inequality $2x^2 + 2x + 1 < 0$ is false for all $x \in R_e$; i.e., $\{x \in R_e \mid 2x^2 + 2x + 1 < 0\} = \emptyset$.

In summary, the quadratic inequality can be studied through the general rules of inequalities as applied to real numbers assisted by factorization, the method of completing squares, or the quadratic formula.

2.12 CONCEPT OF ABSOLUTE VALUE

A convenient method for expressing the distance of a point from zero on the real-number line, without regard to direction, is the use of the concept of absolute value. The absolute value of x, where $x \in R_e$, is written $|x|$ and defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The absolute value of 3, written $|3|$, is 3; and $|-2| = -(-2) = 2$. Similarly, $|-3 + 5| = |2| = 2$, $|-3|^2 = (3)^2 = 9$, and

$$|-3| + |-5| = 3 + 5 = 8$$

Note that the absolute value of a real number is never a negative real number: $|x| > 0$ if $x \neq 0$, and $|x| = 0$ if $x = 0$.

Example 1. On the real-number line the distance between the points 3 and 7 is 4. This result could be obtained from either $|3 - 7|$ or $|7 - 3|$, since $|3 - 7| = |7 - 3|$. Thus, if we examine any two points a and b on the real-number line, then the distance between them, without regard to direction, is given by either $|a - b|$ or $|b - a|$.

Example 2. A study of the set $G = \{a \in R_e \mid |a| \geq 2\}$ reveals that real numbers greater than or equal to 2, or less than or equal to -2, satisfy the inequality $|a| \geq 2$. Two distinct cases arise:

Case 1: If $a \geq 0$, then $|a| = a$ and $a \geq 2$ (Fig. 28).

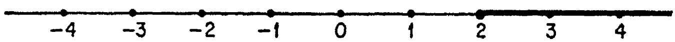


FIG. 28

Case 2: If $a < 0$, then $|a| = -a$ and $-a \geq 2$ or $a \leq -2$ (Fig. 29).

FIG. 29

Since G is the totality of elements satisfying the condition $|a| \geq 2$, then $G = \{a \in R_e \mid a \geq 2 \vee a \leq -2\}$. Set G is represented graphically in Fig. 30.

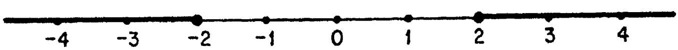


FIG. 30

If a new set $H = \{a \in R_e \mid |a| < 2\}$ is defined, then it may be interpreted graphically by use of Fig. 30. Set H is represented by the interval from -2 to 2 with end points excluded. The two conditions arising from $|a| < 2$, namely, $a < 2$ and $a > -2$, make it possible to describe the designated interval by either $H = \{a \in R_e \mid a < 2 \wedge a > -2\}$ or $H = \{a \in R_e \mid -2 < a < 2\}$.

In general, if c is a positive real number, then $|x| < c$ is equivalent to $x < c \wedge x > -c$ or $-c < x < c$, $|x| > c$ is equivalent to $x > c \vee x < -c$. The alternative way of writing the statement $x < c \wedge x > -c$ in terms of inequality symbols, namely, $-c < x < c$, is not an available procedure for the statement $x > c \vee x < -c$.

In terms of set operations, the solution sets of these defining conditions assume the following forms:

$$\{x \in R_e \mid |x| < c\} = \{x \in R_e \mid x < c\} \cap \{x \in R_e \mid x > -c\}$$
$$\{x \in R_e \mid |x| > c\} = \{x \in R_e \mid x > c\} \cup \{x \in R_e \mid x < -c\}$$

Inequalities of the type $|x - a| < b$, where $b > 0$, are of frequent occurrence in mathematics, and, according to the present discussion, $|x - a| < b$ is equivalent to $-b < x - a < b$. If a is added to each term, then $a - b < x < a + b$. For example, $|x - 2| < 3$ may be written $-3 < x - 2 < 3$ or $-1 < x < 5$. However, $|x - 3| > 2$ must be written $x - 3 > 2 \vee x - 3 < -2$ or $x > 5 \vee x < 1$.

Example 3. The use of the concept of absolute value in defining conditions is illustrated further in the following examples.

a. $H = \{a \in R_e \mid |a| < -2\}$.

No real number will satisfy the condition $|a| < -2$, because the absolute value of any real number a is greater than 0 if $a \neq 0$. Thus, $H = \emptyset$.

b. $T = \{a \in R_e \mid |a - 2| = 3\}$.

If $|a - 2| = 3$, then a is either 5 or -1. These results are obtainable by considering the two cases arising from $|a - 2|$.

Case 1: If $(a - 2) \geq 0$, then $|a - 2| = a - 2$ and $a - 2 = 3$. Thus $a = 5$.

Case 2: If $(a - 2) < 0$, then $|a - 2| = -(a - 2)$ and $-(a - 2) = 3$. Thus $a = -1$.

The graphical interpretation of $T = \{-1,5\}$ is shown in Fig. 31. It is

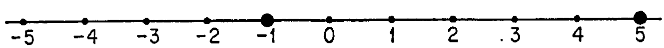


FIG. 31

suggested as an exercise that the solution set for the condition $|2 - a| = 3$ be compared with that of $|a - 2| = 3$.

c. $D = \{x \in R_e \mid |x - 1| < 5\}$.

By use of the definition of absolute value,

$$D = \{x \in R_e \mid x < 6 \wedge x > -4\} \quad \text{or} \quad D = \{x \in R_e \mid -4 < x < 6\}$$

which is translated graphically as shown in Fig. 32. It is left as an exercise to describe $D \cup F$ and $D \cap F$ if $F = \{x \in R_e \mid |x - 1| \geq 5\}$.

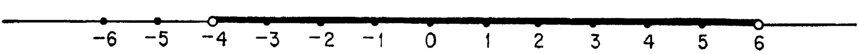


FIG. 32

2.13 INTERVAL NOTATION

The symbol $[a,b]$, where $a < b$ and $a,b \in R_e$, is introduced to mean the interval which includes the end points a and b and all the real numbers between a and b on the real-number line. This notation provides an alternative way of designating the subset of real numbers defined by the inequality $a \leq x \leq b$.

Thus, $\{x \in R_e \mid x \in [a,b]\}$ is accepted to mean the set of all real numbers, including end points, represented by the interval on the real-number line shown in Fig. 33. Since we have occasion frequently to

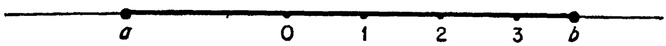


FIG. 33

refer to sets of this kind, they will be denoted in the following manner:

$[a,b] = \{x \in R_e \mid x \in [a,b]\}$	Closed interval from a to b
$[a,b[= \{x \in R_e \mid x \in [a,b[\}$	Half-open interval from a to b, including a but excluding b
$]a,b] = \{x \in R_e \mid x \in]a,b]\}$	Half-open interval from a to b, including b but excluding a
$]a,b[= \{x \in R_e \mid x \in]a,b[\}$	Open interval from a to b, excluding both a and b

For example, if $A =]-2,5[$ and $B = [-7,8]$, then $A \subset B$. The set $[0,\infty[$ defines the set of nonnegative real numbers, while $]-\infty,0[$ defines the set of negative real numbers. The set $]-\infty,\infty[$ describes the set of real numbers in relation to a line extending from $-\infty$ to $+\infty$.

Example 1. The operations of intersection and union may be employed with interval notation.

a. $[3,6] \cap [-2,4] = [3,4]$.

We analyze this graphically as shown in Fig. 34. The interval between the dotted lines represents the intersection set, since both sets have this interval in common.

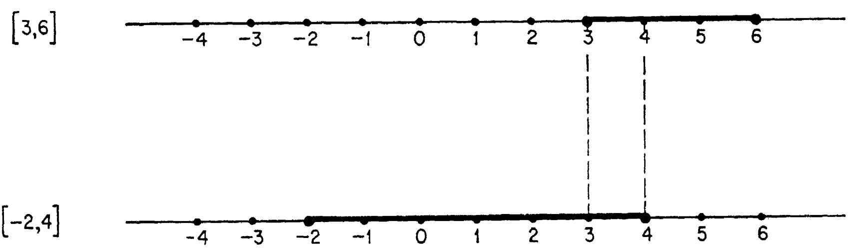


FIG. 34

b. As another example, $]-5,2[\ \cup\ [1,4[\ =\ ]-5,4[$.

Since union implies that a new set is formed which contains either the elements in $]-5,2[$ or those in $[1,4[$ or those in both, we have the result shown in Fig. 35.

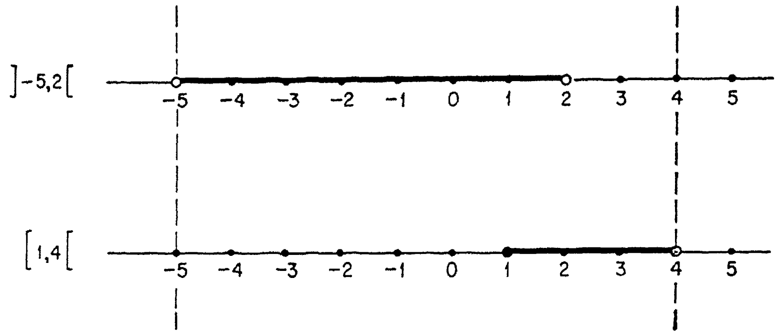


FIG. 35

Exercise 11

1. Which of the following statements are true?

a. $-3 > 0$	*b*. $-5 > -7$	*c*. $\frac{3}{4} > \frac{2}{3}$
d. $-3 < -5$	*e*. $\|-3\| > \|-2\|$	*f*. $\|-5\| + 1 < \|6\|$
g. $\sqrt{5} > 2.236$	*h*. $-5.3851 < -\sqrt{29}$	*i*. $\dfrac{1-\sqrt{3}}{2} > -0.35$
j. $-\frac{7}{3} < -2.33$	*k*. $\|-8\|^2 \leq 64$	*l*. $\|-5\| + \|-3\| > -8$

2. Arrange the following numbers in terms of their absolute values (smallest to largest):

$$-\pi, \frac{3}{\pi}, -\sqrt{2}, 0, 5, 2, -\tfrac{11}{4}$$

3. Evaluate the following:

a. $\|7-3\|$	*b*. $\|6-3\| + \|-5+4\|$
c. $\|-10\| \cdot \|-3\|$	*d*. $\|-8\| - \|-4\|$
e. $\|-3+8\| - \|7\|$	*f*. $\|-12\| + \|3\|$
g. $-\|-4\|^2 + \|-4^3\| - \|-4\|^3$	*h*. $-\|-4^2+12\|^{-2} - \|-4\|^{-1}$

4. Insert an appropriate symbol selected from "=, >, <" between each of the following pairs of numbers:

a. 3, 4	*b*. 0, 5	*c*. -6, 2
d. $\frac{1}{3}, -\frac{3}{4}$	*e*. $-\|-3\|^2, -\|-3^2\|$	*f*. $\pi, \frac{22}{7}$
g. $\|7\|, \|-7\|$	*h*. $-\sqrt{3}, 0$	*i*. $-\sqrt{11}, -\sqrt{13}$
j. $-\|-3\|, \|3\|$	*k*. $\sqrt{4}, 2$	*l*. $\|-5\|, \|-8\|$
m. $\|a\| + \|b\|, \|a+b\|$	*n*. $\|a\| \cdot \|b\|, \|ab\|$	*o*. $\|-8\|^2, \|(-8)^2\|$

5. By use of interval notation, describe each of the following sets. Represent each set graphically. Assume in all cases that $x \in R_e$.

a. $\{x \mid 5x - 7 \geq 8\}$ *b.* $\{x \mid |x| < 4\}$
c. $\{x \mid x^2 - 4x + 3 \leq 0\}$ *d.* $\{x \mid x^2 - 2x + 5 > 0\}$
e. $\{x \mid |x| > -2\}$ *f.* $\{x \mid x^2 + 5x + 6 \geq 0\}$
g. $\{x \mid |x| > 3 \text{ and } |x| < 5\}$ *h.* $\{x \mid |x| < 3 \text{ or } |x| < 8\}$

6. By use of interval notation, find the set defined by each of the following and represent it graphically.

a. $[2,4] \cup [3,11]$ *b.* $[0,3] \cup [-7,1]$ *c.* $[-3,5[\cap]-8,1]$
d. $[6,9] \cap]7,10]$ *e.* $]-\infty,-8[\cup [-5,0[$ *f.* $]-3,0] \cap [1,3[$
g. $]1,5] \cup [1,3]$ *h.* $]-3,-2] \cup]-2,5[$ *i.* $[-5,3] \cap]3,8]$
j. $[-4,3] \cap [2,5[$

7. Discuss the validity of each of the following statements:

a. $[-5,2] \subset]-5,6[$ *b.* $4 \in [0,4]$ *c.* $-6 \in [0,\infty[$
d. $]-\infty,0] \subset]-\infty,\infty[$ *e.* $]-\infty,0] \subset]-\infty,0]$

8. Plot each of the following on a real-number line. Assume in all cases that $x \in R_e$.

a. $\{x \mid 2x + 3 = 3x - 5\}$ *b.* $\{x \mid |x| = 2\}$
c. $\{x \mid |x| \geq 3\}$ *d.* $\{x \mid |x| < -2\}$
e. $\{x \mid |x - 1| < 3\}$ *f.* $\{x \mid x \geq 5 \vee x < -2\}$
g. $\{x \mid 3x < 9\}$ *h.* $\{x \mid x < 2x\}$
i. $\left\{x \,\middle|\, \frac{x}{2} - 3 + \frac{5x - 1}{3} > 0\right\}$ *j.* $\left\{x \,\middle|\, \left|\frac{x^2 - x}{x}\right| \leq 4\right\}$
k. $\{x \mid x \text{ is a prime less than } 20\}$ *l.* $\{x \mid x = 2,4,6, \ldots, 2n, \ldots\}$
m. $\{x \mid 5x \neq 15\}$ *n.* $\{x \mid |x| \leq 4\}$
o. $\{x \mid (x + 2)(x - 1) < (x - 1)^2\}$ *p.* $\{x \mid x^2 - 9 > 0\}$
q. $\{x \mid |x| < 0\}$ *r.* $\{x \mid |x + 2| \leq 5\}$
s. $\left\{x \,\middle|\, \frac{x - 2}{x + 3} < 0\right\}$ *t.* $\left\{x \,\middle|\, \frac{x - 5}{x + 2} > 0\right\}$

9. If $A = \{0,1,2,3,4\}$, tabulate each of the following sets:

a. $\{x \mid x \in A \text{ and } 3x = 4\}$ *b.* $\{x \mid x \in A \wedge 3x > 4\}$
c. $\{x \mid x \in A \text{ and } 3x < 4\}$ *d.* $\{x \in A \mid x^2 - 3x + 2 = 0\}$
e. $\{x \in A \mid x^2 - 2x - 3 = 0\}$ *f.* $\{x \in A \mid x^2 - 2x - 3 > 0\}$
g. $\{x \in A \mid x^2 - 2x - 3 < 0\}$

10. Describe more concisely the defining condition of each of the following sets, and interpret its solution set graphically. Assume in all cases that $x \in R_e$.

a. $\{x \mid 3x - 3 \geq 12\}$ *b.* $\{x \mid 2x^2 - x - 4 \geq 2x^2 + 4x - 5\}$
c. $\{x \mid -2(x - 8) < -3(x + 2)\}$ *d.* $\{x \mid x^2 - 6x + 5 \leq 0\}$
e. $\{x \mid 2x^2 - 5x - 3 > 0\}$ *f.* $\{x \mid x^2 - 6x + 9 < 0\}$
g. $\{x \mid x^2 - 4x + 3 \geq 0\}$ *h.* $\{x \mid x^2 + x + 2 > 0\}$
i. $\{x \mid 3x^2 - 4x - 4 = 0\}$ *j.* $\{x \mid x^2 + x - 2 > 0\}$
k. $\{x \mid x^2 - 2x + 5 < 0\}$

11. Consider the sets (assume $a \neq 0$):

$$S_1 = \{x \mid ax + b = 0\} \qquad S_4 = \{x \mid ax^2 + bx + c = 0\}$$
$$S_2 = \{x \mid ax + b > 0\} \qquad S_5 = \{x \mid ax^2 + bx + c > 0\}$$
$$S_3 = \{x \mid ax + b < 0\} \qquad S_6 = \{x \mid ax^2 + bx + c < 0\}$$

a. Graph each of the solution sets S_1, S_2, S_3, S_4, S_5, and S_6, if given that $x \in R_e$ and the designated values of a, b, and c as they appear in the tables.

a	b	S_1	S_2	S_3
2	3	$\{x \mid 2x + 3 = 0\}$	$\{x \mid 2x + 3 > 0\}$	$\{x \mid 2x + 3 < 0\}$
2	-3			
4	0			
-3	-4			

a	b	c	S_4	S_5	S_6
1	-3	2	$\{x \mid x^2 - 3x + 2 = 0\}$	$\{x \mid x^2 - 3x + 2 > 0\}$	$\{x \mid x^2 - 3 + 2 < 0\}$
1	3	2			
3	2	-1			
3	2	1			

b. In each case compare $S_1 \cup S_2 \cup S_3$ with the real-number line. Can a general conclusion be stated for any choice of a and b? What can correspondingly be said about $S_4 \cup S_5 \cup S_6$?

12. Graph each of the following sets on a real-number line. Assume in all cases that $x \in R_e$.

a. $\{x \mid -1 < x < 3\}$ *b*. $\{x \mid 0 \leq x \leq 5\}$
c. $\{x \mid -5 < x \leq 3\}$ *d*. $\{x \mid x \in [-5,3[\text{ or } x \in [-2,5[\}$
e. $\{x \mid x \in [0, \infty[\text{ and } x \in]-\infty,0[\}$

13. Express the defining conditions in parts *a* to *c* of Problem 12 by using interval notation. Rewrite the defining conditions of part *e* in Problem 12 in the form used in parts *a* to *c*. Can the defining conditions in part *d* be written in the same form as that of part *a*?

14. If $x \in R_e$, graph the following:

a. $\{x \mid x^2 \leq 4\} \cap \{x \mid x \in [-1,2]\}$ *b*. $\{x \mid \sqrt{x} < 1.7\} \cup [-2,1]$
c. $\{x \mid -2 < x \leq 3\} \cap \{x \mid -5 < x \leq 1\}$
d. $\{x \mid x^2 > 4\} \cap \{x \mid x^2 < 16\}$ *e*. $\{x \mid x^3 > 1\} \cap \{x \mid x^3 < 8\}$

15. If $x \in R_e$, $A = \{x \mid |x| \geq 2\}$, $B = \{x \mid |x - 2| \leq -1\}$, $C = \{x \mid x^2 \leq 2\}$, and $D = \{x \mid x^2 - 2x - 8 \leq 0\}$, graph the following:

a. $A \cap B$ *b*. $A \cap C$ *c*. A'
d. $A' \cap B$ *e*. $B' \cap D$ *f*. $A \cup D$
g. $(C \cap D) \cap B$ *h*. $D' \cap A$ *i*. $A' \cap C'$
j. $(A \cup B) \cap (C \cup D')$

3

Ordered Pairs, Cartesian Product, and Relations

3.1 INTRODUCTION

The ability to understand and classify mathematical concepts with respect to some general frame of reference is of utmost importance for mathematical maturity. The majority of ideas in elementary mathematics fall into broad areas, such as sets, numbers, conditions, and relations. In turn, these notions have an effect upon one another as they become interpreted and interwoven in the light of various applications.

This chapter introduces a new type of mathematical object called an ordered pair from which emanates two basic ideas, namely, a cartesian product and a relation. Ordered pairs and the cartesian product provide the frame of reference for a systematic examination of defining conditions and relations. This environment provides added meaning for the concepts of equations and inequalities, since the communication of these ideas becomes more precise through the vocabulary and symbolism of set theory.

3.2 ORDERED PAIRS

A pair of objects, one of which is designated as the first component and the other as the second component, is called an ordered pair. If we write the numbers 2 and 3 in parentheses (2,3), designating 2 as the first component and 3 as the second component, we have an ordered pair. The pair of objects (socks, shoes) needs to be ordered, since in the process of dressing one puts on socks before shoes. In general, (a,b) is referred to as an ordered pair where the arrangement within the parentheses specifies that a is to be taken as the first component and b the second component. It is noted that the components of an ordered pair need not be different; that is, (3,3), (5,5), and (a,a) are valid pairs.

A new kind of element called an "ordered pair" has been developed, and in the material which follows, many interpretations will be attached to this element and its components. For example, the concept of an ordered pair is used in the basic development and extension of number systems in mathematics. The ordered pair (a,b) is frequently used as a representation for the rational number a/b or the complex number $a + bi$. Thus, the set of rational numbers or the set of complex numbers can be defined as the set of all ordered pairs (a,b), where in the first case $a,b \in I$ and $b \neq 0$ and in the second case $a,b \in R_e$. Here the ordered pair (a,b) is used and a meaning attached which is pertinent to a particular objective, that of extending or developing a system of numbers. For each distinct interpretation of an ordered pair, it becomes necessary to define operations, to state laws or postulates which guide these operations, and to give meaning to the equality of ordered pairs.

For purposes which will become apparent, the equality of ordered pairs is defined as follows: Two ordered pairs (a,b) and (c,d), where $a,b,c,d \in R_e$, are equal—$(a,b) = (c,d)$—if and only if $a = c$ and $b = d$. We note that $(a,b) = (b,a)$ if and only if $a = b$.

3.3 CARTESIAN-PRODUCT SET

The construction of new sets from a given set was illustrated through subset construction and power-set construction in Sections 1.7 and 1.8. A third kind called "cartesian-set construction" is now introduced, for which the basic elements are ordered pairs. Suppose the set $A = \{2,3,4\}$ is considered. It is now possible to form nine ordered pairs by using the elements of A. For example, if the element 2 is associated with each and every element of A, then the ordered pairs (2,2), (2,3), and (2,4) are obtained. The following scheme enables us to perform all the pairings in a systematic fashion:

$$\begin{tikzcd}
 & 2 \arrow[r] & (2,2) & & 2 \arrow[r] & (3,2) & & 2 \arrow[r] & (4,2) \\
2 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (2,3) & 3 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (3,3) & 4 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (4,3) \\
 & 4 \arrow[r] & (2,4) & & 4 \arrow[r] & (3,4) & & 4 \arrow[r] & (4,4)
\end{tikzcd}$$

In this manner, nine ordered pairs are formed where all the components of the ordered pairs are elements of A. This totality of ordered pairs is called the "cartesian set of A" and designated $A \times A$. Note that the elements of A are single numbers, while the elements of $A \times A$ are ordered pairs. By use of the tabulation method,

$$A \times A = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$$

Hence, the cartesian set of A is the set $A \times A$ of ordered pairs (a,b)

where $a \in A$ and $b \in A$. If A is a unit set—$A = \{a\}$—then

$$A \times A = \{(a,a)\}$$

If A is a finite set containing n elements, then $A \times A$ is a finite set containing n^2 ordered pairs of elements. When A is an infinite set, then $A \times A$ is also an infinite set.

$A \times A$ can be interpreted graphically by drawing two perpendicular lines with assigned scales so as to associate points on each line with the elements of A. For the example $A = \{2,3,4\}$, the graph of $A \times A$ is as shown in Fig. 36. The point in the plane associated with the ordered pair (2,3) is located at the intersection of the vertical line through 2 and the horizontal line through 3. Similarly, the other elements of $A \times A$ are associated with points of the plane, yielding what is called a "nine-point lattice," the graph of $A \times A$.

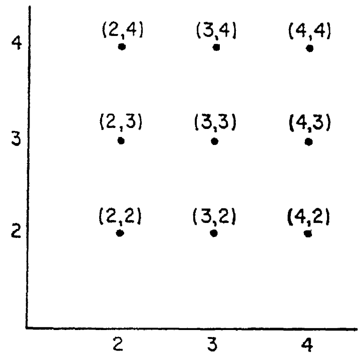
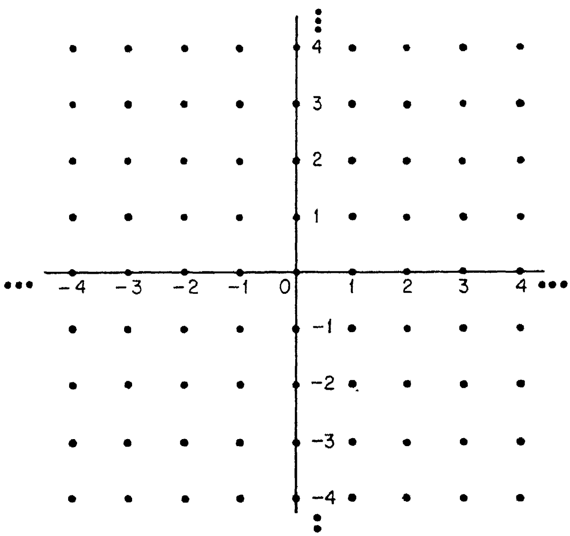


FIG. 36

If I = set of integers, then $I \times I$ contains an infinite number of ordered pairs of integers. As a result, $I \times I$ cannot be represented completely through either a tabulation procedure or a graph. However, the incomplete lattice of $I \times I$ (Fig. 37) indicates how the ordered pairs



INCOMPLETE GRAPH OF $I \times I$

FIG. 37

appear graphically and also indicates that a 1-1 correspondence exists between the lattice points and the set of ordered pairs (a,b) where $a \in I$ and $b \in I$.

Thus far the discussion has been confined to a single set of elements in the formation of a cartesian-product set. However, this same procedure may be extended to involve the elements of any two sets A and B.

Example 1. If $A = \{2,4,6\}$ and $B = \{3,5\}$, then the cartesian-product set, denoted $A \times B$, is the set of all ordered pairs whose first components are chosen from A and whose second components are chosen from B. The scheme for the formation of all the possible ordered pairs yields

$$\begin{tikzcd}[row sep=small]
 & 3 \arrow[r] & (2,3) & & 3 \arrow[r] & (4,3) & & 3 \arrow[r] & (6,3) \\
2 \arrow[ur] \arrow[dr] & & & 4 \arrow[ur] \arrow[dr] & & & 6 \arrow[ur] \arrow[dr] & & \\
 & 5 \arrow[r] & (2,5) & & 5 \arrow[r] & (4,5) & & 5 \arrow[r] & (6,5)
\end{tikzcd}$$

$A \times B = \{(2,3),(2,5),(4,3),(4,5),(6,3),(6,5)\}$. If it is desired to form $B \times A$, then the first components of the ordered pairs are chosen from B and the second components from A. The formation $B \times A$ yields

$$\begin{tikzcd}[row sep=small]
 & 2 \arrow[r] & (3,2) & & 2 \arrow[r] & (5,2) \\
3 \arrow[ur] \arrow[r] \arrow[dr] & 4 \arrow[r] & (3,4) & 5 \arrow[ur] \arrow[r] \arrow[dr] & 4 \arrow[r] & (5,4) \\
 & 6 \arrow[r] & (3,6) & & 6 \arrow[r] & (5,6)
\end{tikzcd}$$

$B \times A = \{(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$. Here $A \times B \neq B \times A$, since the elements of $A \times B$ may in general be different from those of $B \times A$.

In summary, $A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$. The cartesian set $A \times A$ is a special case of the cartesian product $A \times B$ when $A = B$. However, $A \times B \neq B \times A$ when $A \neq B$.

If in Example 1 the number of ordered pairs on $A \times B$ and $B \times A$ is denoted by $n(A \times B)$ and $n(B \times A)$, respectively, then

$$n(A \times B) = n(B \times A) = 6$$

Accordingly, A contains three elements and B contains two elements, from which follows $n(A) \cdot n(B) = 3 \cdot 2 = 6$. In general, if A and B are finite sets, A with $n(A)$ elements and B with $n(B)$ elements, then $A \times B$ contains $n(A \times B)$ elements and $n(A \times B) = n(A) \cdot n(B)$. Here $A \times B$ and $B \times A$ are not equal, but they are equivalent, since a 1-1 correspondence can be established between the elements of these two cartesian products.

Example 2. If $A = \{1,2,3,4\}$ and $C = \{1,2,3\}$, then

$$A \times C = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)\}; n(A) = 4; n(C) = 3; \text{and } n(A \times C) = n(A) \cdot n(C) = 12.$$

$$C \times A = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}; n(C) = 3; n(A) = 4; \text{and } n(C \times A) = n(C) \cdot n(A) = 12.$$

Note in Fig. 38 that the horizontal line is associated in each case with that set from which the first components are selected. The graphs in the figure indicate that $A \times C$ and $C \times A$ do not represent the same set of points in the plane.

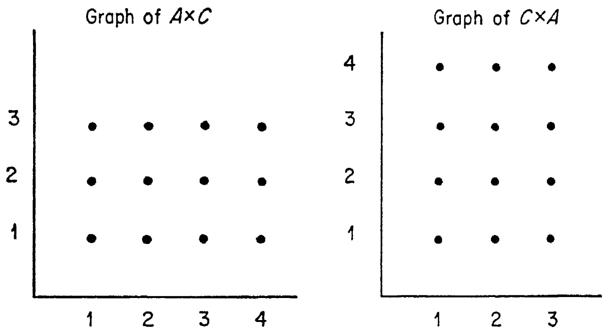


FIG. 38

It was stated earlier that all real numbers may be represented geometrically on a coordinate line where a 1-1 correspondence can be established between the set of points on the line and the set of real numbers. This line is usually drawn in either a horizontal or a vertical position. If now two coordinate lines are introduced in a plane so that they are perpendicular with their zero points coinciding, there is formed what is referred to as a cartesian (rectangular) coordinate system. Each point P in the plane can now be associated with an ordered pair of real numbers (x,y) where the first component x is the measure of the directed distance of P from the vertical coordinate line, while the second component y is the measure of the directed distance of P from the horizontal coordinate line. The components of the ordered pair (x,y) are called the coordinates of P; the first component is the x coordinate (abscissa) and the second component is the y coordinate (ordinate). Since x and y are commonly used to designate the coordinates of a point P, the two perpendicular lines are referred to as the x axis (horizontal line) and the y axis (vertical line), and their intersection point (0,0) is called the origin. The entire plane is then described as the xy plane and is a graphical interpretation of the totality of all ordered pairs (x,y) arising from the cartesian product $R_e \times R_e = \{(x,y) \mid x \in R_e \wedge y \in R_e\}$.

A 1-1 correspondence can now be established between the set of points of the plane and the set of ordered pairs of real numbers by associating each point in the plane with an ordered pair of real numbers and, conversely, associating each ordered pair of real numbers with a point in the plane.

The concept of cartesian product can be extended to more than two sets. For example, the cartesian-product set $A \times B \times C$ can be defined as the set of all ordered triples (a,b,c) where $a \in A$, $b \in B$, and $c \in C$. If $A = \{1,2\}$, $B = \{3,4\}$, and $C = \{2,3,5\}$, then

$$A \times B \times C = \{(1,3,2),(1,3,5),(1,3,3),(1,4,2),(1,4,3),(1,4,5),(2,3,2),\\(2,3,3),(2,3,5),(2,4,2),(2,4,3),(2,4,5)\}$$

Schematically, $A \times B \times C$ could be obtained as follows:

$$\begin{tikzcd}[row sep=small]
 & & 2 \arrow[r] & (1,3,2) \\
 & 3 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (1,3,3) \\
1 \arrow[ur] \arrow[ddr] & & 5 \arrow[r] & (1,3,5) \\
 & & 2 \arrow[r] & (1,4,2) \\
 & 4 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (1,4,3) \\
 & & 5 \arrow[r] & (1,4,5)
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & & 2 \arrow[r] & (2,3,2) \\
 & 3 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (2,3,3) \\
2 \arrow[ur] \arrow[ddr] & & 5 \arrow[r] & (2,3,5) \\
 & & 2 \arrow[r] & (2,4,2) \\
 & 4 \arrow[ur] \arrow[r] \arrow[dr] & 3 \arrow[r] & (2,4,3) \\
 & & 5 \arrow[r] & (2,4,5)
\end{tikzcd}$$

The set $R_e \times R_e \times R_e$ defines the set of all ordered triples of real numbers which, in turn, may be associated with the points of a three-dimensional space.

Exercise 12

1. If $U = \{1,2,3,4,5\}$, $A = \{3,4,5\}$, and $B = \{1,2,3,4\}$, use the tabulation method to describe each of the following sets:

a. $A \times A$ *b.* $A \times B$ *c.* $B \times B$
d. $B \times A$ *e.* $U \times U$ *f.* $(A \times A) \cap (B \times B)$
g. $(A \times A) \cup (B \times B)$ *h.* $(A \times B) \cup (B \times A)$ *i.* $(A \times B) \cap (B \times A)$
j. $(U \times A) \cap (U \times B)$

2. Let C be a set with n elements and B a set with m elements. How many elements are contained in each of the following sets?

a. $C \times C$ *b.* $B \times B$
c. $B \times C$ *d.* $B \times B \times B$
e. $\{(x,y) \mid x \in C \wedge y \in C \wedge x = y\}$ *f.* $\{(x,y) \mid x \in B \wedge y \in B \wedge x \neq y\}$

3. Given $N = \{\text{natural numbers}\}$, $I^+ = \{\text{positive integers}\}$, list 10 ordered pairs in $N \times I^+$. How is $I^+ \times N$ related to $N \times I^+$?

3.4 RELATIONS

Frequently statements of the type "John is married to Sue," "7 is less than 9," "Peg is the sister of Bob," and "Texas is larger than Ohio" are heard in everyday conversation. Each of these sentences involves what is intuitively understood to be a relationship, and expressions of the type "is married to," "is equal to," "is the father of," "is a multiple of," "is a member of," and "is included in" are classified as connectives that yield relations. The word "relation" implies a correspondence or an association of two objects (people, numbers, ideas, etc.) according to some property possessed by them. Suppose we consider the sentence

$$x \text{ } was\ the\ father\ of\ y$$

The ordered pair (David, Solomon) will satisfy the sentence if x is replaced by David and y is replaced by Solomon. However, the ordered pair (Solomon, David) does not satisfy the sentence. Similarly, the ordered pairs (3,4), (2,5), and (6,8) satisfy the sentence x *is less than* y, while (4,3), (5,2), and (3,1) do not. If a sentence is given in the form x ____________ y, with the blank to be filled by some connective expression, then some universe must be designated from which ordered pairs are to be selected for testing in the sentence. In general, a partition of two sets of ordered pairs is formed: the set of those ordered pairs which satisfy the sentence and the set of those which do not. The question arises as to whether the relation is the original "connective expression" or "the set of ordered pairs satisfying the sentence." From a mathematical standpoint the meaning of a relation is more precise if it is defined to be a set of ordered pairs. Thus if R represents a relation, a set of ordered pairs, and (x,y) satisfies the connective property, we write $(x,y) \in R$. This is sometimes represented as xRy.

Example 1. Consider the set $U = \{1,2,3\}$ and select those ordered pairs from $U \times U$ which are elements of $R = \{(x,y) \mid x \text{ is greater than } y\}$. Here if the connective expression "is greater than" is replaced by the mathematical symbol "$>$," then $R = \{(x,y) \mid x > y\}$, where $x \in U$ and $y \in U$. If xRy is used in place of $(x,y) \in R$, then a resemblance between the two sentences xRy and $x > y$ is evident. Accordingly, it is agreed that the symbols "R" and "$>$" are interchangeable. Hence, the set $\{(x,y) \mid x > y\}$ may be named and symbolized by either R or $>$. Thus,

$$R = \{(x,y) \mid x > y\}$$

can be written either

$$> \; = \{(x,y) \mid x \text{ is greater than } y\}$$

or

$$> \; = \{(2,1),(3,1),(3,2)\}$$

It follows that $(2,1) \in R$ or $(2,1) \in >$. The use of "$>$" as a name for the tabulated set $\{(2,1),(3,1),(3,2)\}$ is more informative as to how the relation was actually formed. However, when a condition, such as $x > y$, appears within the braces—$\{(x,y) \mid x > y\}$—it is more convenient to name the relation by some letter, such as R, rather than by $>$. The notation $(x,y) \in R$ or xRy is used in preference to $(x,y) \in >$. If an element (x,y) does not belong to R, we write $(x,y) \notin R$ or $x \not R y$.

Consequently, expressions of the type "$>$," "$=$," "is the brother of," "is an element of," and "$\subseteq$" will be referred to as relations. This makes possible the consideration of both the connective expression and the set of ordered pairs as one entity, the relation, where it is imperative that whenever connective expressions are involved, they are meaningful only if associated with some specified universe. If the ordered pairs are extracted from a universe $U \times U$ in the formation of a particular relation R, the phraseology "a relation R on a set U" is used to describe it.

Example 2. If the connective "is a factor of" on $U = \{2,3,4,5,6\}$ is considered, the relation R is formed, namely, $R = \{(x,y) \mid x \text{ is a factor of } y\}$, where $(x,y) \in U \times U$. A tabulation for this relation R is $\{(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(5,5),(6,6)\}$. It should be observed that $R \subset U \times U$.

To further clarify the meaning of a relation, the following example is included.

Example 3. Let $A = \{2,3\}$ and $B = \{1,2\}$. Then

$$A \times B = \{(2,1),(2,2),(3,1),(3,2)\}$$

From the set $A \times B$ it is possible to construct 16 different subsets, which are listed in Table 1.

Table 1

Four elements	Three elements	Two elements	One element	No element
$\{(2,1),(2,2),(3,1),(3,2)\}$	$\{(2,1),(2,2),(3,1)\}$	$\{(2,1),(2,2)\}$	$\{(2,1)\}$	$\{\ \}$
	$\{(2,1),(2,2),(3,2)\}$	$\{(2,1),(3,1)\}$	$\{(2,2)\}$	
	$\{(2,1),(3,1),(3,2)\}$	$\{(2,1),(3,2)\}$	$\{(3,1)\}$	
	$\{(2,2),(3,1),(3,2)\}$	$\{(2,2),(3,1)\}$	$\{(3,2)\}$	
		$\{(2,2),(3,2)\}$		
		$\{(3,1),(3,2)\}$		

If one of these subsets is considered, say $\{(2,1),(3,1),(3,2)\}$, and is designated by R_1, then R_1 is being described by the tabulation method. To specify R_1 by the defining-property method requires a set description such as $\{(x,y) \mid \text{some defining condition about } x \text{ and } y\}$ or $\{(x,y) \mid P_{xy}\}$, where P_{xy} represents the defining condition in x and y. Thus R_1 may

be written in the following manner: $\{(x,y) \mid x > y\}$, where $x \in A$, $y \in B$, and P_{xy} is replaced by $x > y$. If x and y are replaced in the statement $x > y$ by the corresponding components of each of the ordered pairs belonging to R_1, a true statement is obtained. For example, the ordered pair (3,2) yields the true statement $3 > 2$ when x is replaced by 3 and y by 2. Though there exist other subsets of $A \times B$, such as $R_2 = \{(2,1),(3,2)\}$, each of whose ordered pairs satisfy $x > y$, R_1 is the subset which contains the largest number of ordered pairs from $A \times B$ that satisfies the given condition. Hence, if the solution set of $x > y$ with respect to $A \times B$ is required, R_1 must be chosen in preference to R_2. Other defining conditions may be utilized which generate the same solution set R_1 from $A \times B$. For example, $\{(x,y) \mid y < x\}$, or $\{(x,y) \mid y \neq x\}$, or $\{(x,y) \mid y \leq x - 1\}$ yields the solution set designated as R_1 with respect to $A \times B$.

There are many ways of describing the subsets of $A \times B$ by the defining-property method. The subsets $R_2 = \{(2,1),(3,2)\}$ and $R_3 = \{(2,2)\}$ may be described as $\{(x,y) \mid y = x - 1\}$ and $\{(x,y) \mid y = x\}$, respectively. Even though a specific subset of $A \times B$ may have different descriptions, the graphical interpretation depends upon the ordered pairs exposed in a corresponding tabulation. The graphs of $A \times B$, R_1, and R_2 appear in Fig. 39.

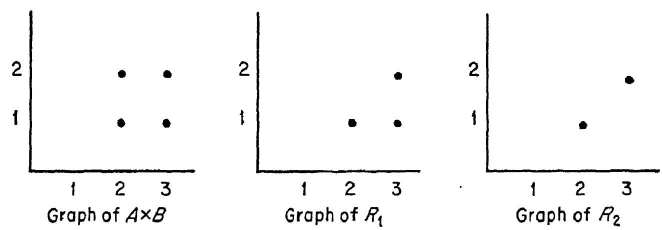


FIG. 39

Each of the subsets listed in Table 1 is a relation, a set of ordered pairs, chosen from the cartesian product $A \times B$. Accordingly, we refer to a subset of ordered pairs as a "relation in a cartesian-product set." Thus R_1, R_2, and R_3 are relations in $A \times B$. Further, the universal set $A \times B$ and the empty set $\emptyset$ are also considered relations in $A \times B$, since $A \times B \subseteq A \times B$ and $\emptyset \subseteq A \times B$. These special sets are commonly referred to by their familiar names, the universe and the null set.

If R is a relation in $A \times B$, then the set of first components of all the ordered pairs of R constitutes the "domain of definition of R" or "domain of R" and is written as D^*. The set of second components is called the "range of values of R" or "range of R" and written as R^*. For example, if $R = \{(3,4),(0,6),(2,1),(8,7),(3,3)\}$, then domain $D^* = \{3,0,2,8\}$, while range $R^* = \{4,6,1,7,3\}$. We note that $D^* \subseteq A$ and $R^* \subseteq B$.

Any set of ordered pairs is a relation. These pairs may come from various environments such as mathematical tables, charts, graphs, word descriptions, and symbolic statements. For example, a table of logarithms yields pairs of the type $(x, \log_a x)$, while pairs such as (time, temperature) can be obtained from an hourly-temperature graph.

3.5 EQUIVALENCE RELATIONS

A relation R on a set U is called an "equivalence relation" if it possesses the three significant properties of reflexivity, symmetry, and transitivity.

Given that R is a relation on a set U, then:

a. R is said to be "reflexive" if for each a in U, $(a,a) \in R$ or aRa. For example, Tom may be the brother of Joe, but Tom is not the brother of Tom. Hence, the relation "is the brother of" is not reflexive, since (Tom, Tom) does not belong to the relation. The relation "is as tall as" is reflexive, since we know that "Tom is as tall as Tom." If $A = \{1,2,3,4\}$, then the relation "$=$" extracts from $A \times A$ the ordered pairs (1,1), (2,2), (3,3), (4,4). It follows that "$=$" is a reflexive relation.

b. R is said to be symmetric if whenever $(a,b) \in R$, then $(b,a) \in R$. This represents a reversible property in that aRb implies bRa. The relation "is married to" is a symmetric relation. If "Tom is married to Jean," then it follows that "Jean is married to Tom." Hence both (Tom, Jean) and (Jean, Tom) belong to the relation. Other symmetric relations are "is not married to," "is as old as," "is the cousin of," "is equal to," "is parallel to," and "is similar to." Examples of relations that are not symmetric are "is less than," "is the father of," "is an element of," and "is contained in."

c. R is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then it follows that $(a,c) \in R$. This, expressed in the other notation, states that if aRb and bRc, then aRc. Examples of transitive relations are "is equal to," "is a factor of," "is a multiple of," "is greater than," and "is less than"; while "is the father of" and "is in love with" are not transitive relations.

Throughout this text, relations such as "$\in$," "$\subseteq$," "$\subset$," and "$=$" are used with sets; and "$<$," "$=$," "$>$," "$\leq$," and "$\geq$" are used with numbers. Certain of these relations are now examined so as to determine whether they are equivalence relations.

Example 1. Consider the relation "$=$" with respect to sets.

Reflexive:	If A is a set, then $A = A$.	True
Symmetric:	If A and B are sets and $A = B$, then $B = A$.	True
Transitive:	If A, B, and C are sets and if $A = B$ and $B = C$, then $A = C$.	True

Hence "$=$" is an equivalence relation.

Example 2. The relation "$\subset$" with respect to sets does not possess all three properties.

Reflexive: If A is a set, then $A \subset A$. False
Symmetric: If A and B are sets and $A \subset B$, then $B \subset A$. False
Transitive: If A, B, and C are sets and if $A \subset B$ and $B \subset C$, then $A \subset C$. True

Hence "$\subset$" is not an equivalence relation, since it possesses only the transitive property.

Example 3. Let the relation "is the mother of" be considered on the set of all people.

Reflexive: a "is the mother of" a. False
Symmetric: If a "is the mother of" b, then b "is the mother of" a. False
Transitive: If a "is the mother of" b, and b "is the mother of" c, then a "is the mother of" c. False

Hence this relation does not possess any of the three properties.

Example 4. Let the relation "$\cong$," congruence, be considered on all triangles in euclidean plane geometry.

Reflexive: $A \cong A$. True
Symmetric: If $A \cong B$, then $B \cong A$. True
Transitive: If $A \cong B$ and $B \cong C$, then $A \cong C$. True

The relation "$\cong$" is an equivalence relation.

Example 5. Let the relation "$\geq$" be considered on all numbers in the set $\{a \mid a \in R_e\}$.

Reflexive: $a \geq a$. True, since $a = a$
Symmetric: If $a \geq b$, then $b \geq a$. This statement is true for the case $a = b$, but false in all other instances.
Transitive: If $a \geq b$ and $b \geq c$, then $a \geq c$. True

The relation "$\geq$" is not an equivalence relation, since it lacks the property of symmetry.

Exercise 13

1. Given $U = \{2,3,4,5,6\}$, describe each of the following relations on U by both the tabulation and the defining-property method. Determine the domain D^* and the range R^* for each of these relations.

a. "is equal to"
b. "is a factor of"
c. "is a multiple of"
d. "is less than"
e. "is greater than"
f. "is equal to or greater than"
g. "is not equal to"
h. "is 2 greater than"

2. Graph each of the relations in Problem 1.

3. Which of the following are equivalence relations?

	Relation	*Universe*
a.	"is the brother of"	People
b.	"is the same weight as"	People
c.	"is at peace with"	Countries
d.	"is the husband of"	People
e.	"is parallel to"	Lines in plane geometry
f.	"is perpendicular to"	Lines in plane geometry
g.	"has the same area as"	Polygons
h.	"is afraid of"	People
i.	"has the same color hair as"	People
j.	"is an element of"	Sets
k.	"is the supplement of"	Angles in plane geometry
l.	"$<$"	Real numbers
m.	"is disjoint from"	Sets
n.	"is equivalent to"	Sets

4. Let $A = \{1,2,3,4,5,6\}$ and $B = \{1,2,3,4\}$. Graph each of the following relations in the cartesian-product set $A \times B$. Determine D^* and R^* for each of these relations.

a. "is equal to"
b. "is twice"
c. "is a divisor of"
d. "is 3 less than"
e. "is three times ________ less 3"
f. "forms a proper fraction with"

5. In Table 1, Section 3.4, omitting U and $\emptyset$, examine the remaining subsets and describe as many of these as possible by defining conditions.

6. For each of the following, give an example of a relation that possesses the stated properties:

a. Reflexive, symmetric, and transitive
b. Reflexive and symmetric, but not transitive
c. Not reflexive, not symmetric, and not transitive
d. Not reflexive and not symmetric, but transitive
e. Reflexive, not symmetric, but transitive

3.6 COMPLEMENTARY RELATIONS

It was previously stated that if a set A is a subset of some universe U, there exists another set called the complement of A, written A', where A' contains those elements of U not in A. Similarly, a relation R in a cartesian-product set $U \times U$ (or a relation R on U) has a complement R' in $U \times U$. This set R' contains as elements those ordered pairs of $U \times U$ not in R. Hence it follows that $R' \cup R = U \times U$. Since R' is a subset of ordered pairs, R' is referred to as the "complementary relation" of R. Thus, "is the father of" and "is not the father of" are complementary relations.

Example 1. If $U = \{1,2,3\}$ and $R = \{(x,y) \mid y = x\}$, then

$$\begin{aligned} U \times U &= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\} \\ R &= \{(1,1),(2,2),(3,3)\} \\ R' &= \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\} = \{(x,y) \mid y \neq x\} \end{aligned}$$

The graphs of $U \times U$, R, and R' appear in Fig. 40.

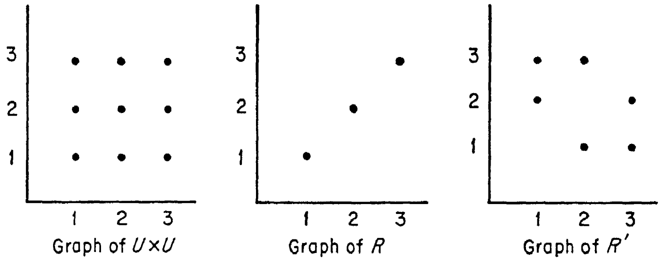


FIG. 40

Example 2. Additional examples of complementary relations are:

	Relation	*Complementary relation*
a.	"is equal to"	"is not equal to"
b.	"is parallel to"	"is not parallel to"
c.	"is not a multiple of"	"is a multiple of"
d.	"is not the mother of"	"is the mother of"
e.	"is greater than"	"is not greater than"

In summary, if R is a relation in cartesian-product set $U \times U$, then $R' = \{(x,y) \mid (x,y) \in U \times U \wedge (x,y) \notin R\}$ is the complementary relation to R. In words, R' is the set containing those ordered pairs (x,y) of $U \times U$ not in R.

3.7 INVERSE RELATIONS

A special type of relation that has extensive use is the "inverse relation." This concept is illustrated in the following examples.

Example 1. If the relation $R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$ in the set $U \times U$, where $U = \{1,2,3,4\}$ is given, then the domain of R is $\{1,2,3\}$ and the range of R is $\{2,3,4\}$. If now the components of all the ordered pairs of R are interchanged, the set $\{(2,1),(3,1),(4,1),(3,2),(4,2),(4,3)\}$ is obtained. This set is designated as R^{-1} and is called the "inverse relation" of R. The domain of R^{-1} is $\{2,3,4\}$ and the range of R^{-1} is $\{1,2,3\}$. It is observed that, as a result of the process of interchanging

the components of the ordered pairs, the domain of R is the range of R^{-1} and the range of R is the domain of R^{-1}.

Graphically, if the points of R are designated by open dots and those of R^{-1} by black dots, we obtain the results shown in Fig. 41. Note that

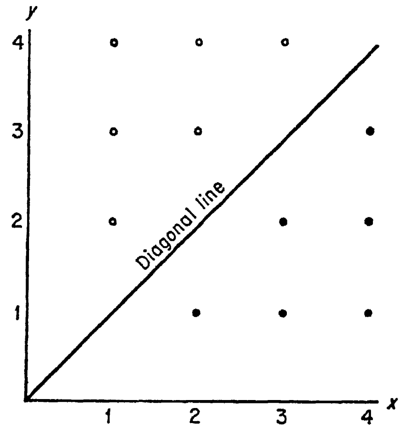


FIG. 41

each point in the graph of R has a corresponding point in the graph of R^{-1}; i.e., if we consider each of the points of R with respect to the diagonal line, there exists a point of R^{-1} that is in balance with it. For example, there exists a correspondence between the points at (1,2) and (2,1), between (2,3) and (3,2), etc. If the diagonal line is considered to be a mirror, each point in R has a reflection or image in R^{-1} and vice versa. For example, the image of (4,3) is (3,4).

The relations R and R^{-1} of Example 1 can be described by the defining-property method. Here $R = \{(x,y) \mid y > x\}$ and $R^{-1} = \{(x,y) \mid x > y\}$. The condition $x > y$, defining R^{-1}, is obtained from the condition $y > x$, defining R, by interchanging the variables. If a relation is represented by some defining condition, the corresponding condition defining the inverse relation is obtained by replacing x for y and y for x. This process yields mirror images of the original points and creates correspondingly a set of ordered pairs which is the "inverse relation."

Example 2. If $R = \{(x,y) \mid x = y + 2\}$ and $S = \{(x,y) \mid y = x^2\}$, then $R^{-1} = \{(x,y) \mid y = x + 2\}$ and $S^{-1} = \{(x,y) \mid x = y^2\}$.

Example 3. If $U = \{-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6\}$ and

$$R = \{(x,y) \mid y = x^2 + 1\}$$

then $R^{-1} = \{(x,y) \mid x = y^2 + 1\}$. If we use black dots for R and open dots for R^{-1}, their respective graphs are as shown in Fig. 42. The points

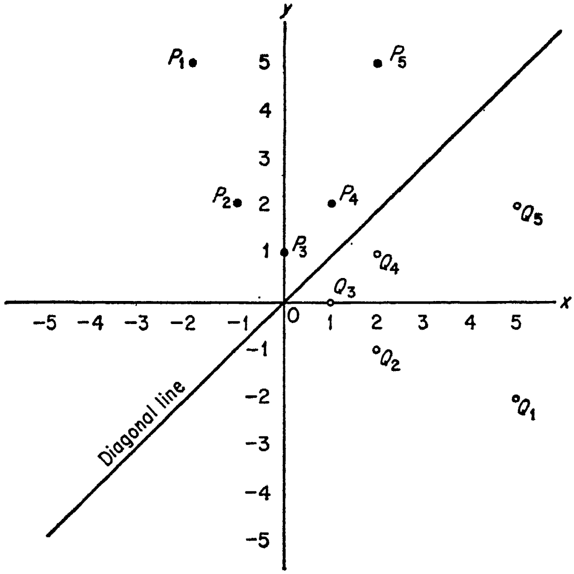


FIG. 42

of one graph are mirror images for those of the other with respect to the diagonal line. For example, Q_1 is the mirror image of P_1 and Q_2 is the mirror image of P_2. The graphs of R and R^{-1} represent isolated points and not continuous curves, since $U \times U \neq R_e \times R_e$.

Example 4. Additional examples of relations and their inverses are:

	Relation	*Inverse*
a.	"is the wife of"	"is the husband of"
b.	"is greater than"	"is less than"
c.	"is a factor of"	"is a multiple of"
d.	"is the student of"	"is the teacher of"
e.	"is three times"	"is one-third of"
f.	"is one-fifth of"	"is five times"
g.	"is 2 less than"	"is 2 more than"
h.	"is 2 more than one-third of"	"is 6 less than three times"

In summary, two relations R and R^{-1} on a cartesian-product set $U \times U$ are inverses of each other if when $(x,y) \in R$, then $(y,x) \in R^{-1}$ and if when $(y,x) \in R$, then $(x,y) \in R^{-1}$.

It is possible for a relation to be its own inverse. For example, if U = the set of real numbers, then the relations $R_1 = \{(x,y) \mid y = x\}$, $R_2 = \{(x,y) \mid xy = 6\}$, and $R_3 = \{(x,y) \mid x^2 + y^2 = 25\}$ are their own inverses. These relations possess the property of symmetry and, when graphed, exhibit this characteristic with respect to a diagonal line used as a mirror line.

Exercise 14

1. If $U = \{-2,-1,0,1,2,3\}$, tabulate and graph each relation R, its inverse R^{-1}, and its complement R'. Specify the domain and range for R, R^{-1}, and R'.

a. $R = \{(x,y) \mid y = 2x\}$
b. $R = \{(x,y) \mid xy = 2\}$
c. $R = \{(x,y) \mid y = 3x - 3\}$
d. $R = \{(x,y) \mid y > x\}$
e. $R = \{(x,y) \mid y - x = 1\}$
f. $R = \{(x,y) \mid x^2 + y^2 = 5\}$
g. $R = \{(x,y) \mid x^2 - y^2 = 3\}$

2. For each R in Problem 1:
(1) Describe R^{-1} by the defining-property method.
Example. If $R = \{(x,y) \mid y = 2x\}$, then $R^{-1} = \{(x,y) \mid x = 2y\}$.
(2) Determine whether $R = R^{-1}$.

3. Complete the following table:

Relation	Complement	Relation	Inverse
$=$		$=$	
$\subset$		$\subset$	
$\in$		$>$	
$\subseteq$		"is the son of"	
"is married to"		"is heavier than"	
"is a factor of"		"is married to"	
"is perpendicular to"		"is a divisor of"	

4*a*. Is the inverse of a complement of a relation identical to the complement of the inverse of the relation? Develop a suitable example and illustrate your conclusion graphically.

b. Since the cartesian product $A \times A$ is a relation, what is its inverse? What is the complement of $A \times A$?

c. Since the empty set $\emptyset$ is a relation, what is its inverse? What is the complement of $\emptyset$?

5. The operations of union and intersection on the relations R_1 and R_2 and on their complements and inverses produce new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $(R_1 \cup R_2)'$, and $(R_1 \cap R_2)'$.

Example. If $U = \{1,2,3\}$, then $U \times U = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$. Let $R_1 = \{(x,y) \mid x \neq y\}$ and $R_2 = \{(x,y) \mid x + y \neq 3\}$. Hence,

$$\begin{aligned}
R_1 &= \{(1,2),(1,3)(2,1)(2,3)(3,1)(3,2)\}\\
R_2 &= \{(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}\\
R_1' &= \{(1,1),(2,2),(2,3)\}\\
R_2' &= \{(1,2),(2,1)\}\\
R_1 \cap R_2 &= \{(1,3),(2,3),(3,1),(3,2)\}\\
R_1 \cup R_2 &= U \times U\\
(R_1 \cap R_2)' &= \{(1,1),(1,2),(2,1)(2,2),(3,3)\}\\
(R_1 \cup R_2)' &= \emptyset\\
R_1^{-1} \cap R_2^{-1} &= R_1 \cap R_2
\end{aligned}$$

Let $U = \{-1,0,1\}$. Graph $R_1 \cup R_2$ and $R_1 \cap R_2$ for each of the following:

a. $R_1 = \{(x,y) \mid y = x\}$
$R_2 = \{(x,y) \mid y < -x\}$

b. $R_1 = \{(x,y) \mid x = y - 1\}$
$R_2 = \{(x,y) \mid x + y = -1\}$

c. $R_1 = \{(x,y) \mid x + y = 2\}$
$R_2 = \{(x,y) \mid y = x^2 + 1\}$

d. $R_1 = \{(x,y) \mid y = x^2\}$
$R_2 = \{(x,y) \mid x + y \geq 0\}$

e. $R_1 = \{(x,y) \mid xy \geq 0\}$
$R_2 = \{(x,y) \mid x + y \leq 0\}$

6. For each of the parts of Problem 5, show that $(R_1 \cup R_2)' = R_1' \cap R_2'$ and $(R_1 \cap R_2)' = R_1' \cup R_2'$.

7. If $U = \{1,2,3\}$, $R_1 = \{(x,y) \mid x \neq y\}$, $R_2 = \{(x,y) \mid x + y \neq 3\}$, and $R_3 = \{(x,y) \mid x + y \geq 3\}$, show that the following statements are true:

a. $R_1 \cup (R_2 \cap R_3) = (R_1 \cup R_2) \cap (R_1 \cup R_3)$
b. $R_1^{-1} \cap (R_2^{-1} \cup R_3^{-1}) = (R_1^{-1} \cap R_2^{-1}) \cup (R_1^{-1} \cap R_3^{-1})$
c. $(R_2')' = R_2$

PROJECTS

Supplementary Exercises

1. By the use of the definition of $a < b$ and the properties of real numbers (Section 2.7), prove each of the following:

a. If $a < b$ and $c > 0$, then $ac < bc$.
b. If $a < b$ and $c < 0$, then $ac > bc$.
c. If $ab > 0$, then either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.
d. If $ab < 0$, then either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

2. If $abc > 0$, where $a,b,c \in R_e$, then it follows that either all three factors are greater than zero or exactly one of the factors is greater than zero, while the other two are less than zero.

Example. If $G = \{x \in R_e \mid (x - 1)(x + 2)(x - 5) > 0\}$, then G can be expressed as $G = G_1 \cup G_2 \cup G_3 \cup G_4$, where

$$\begin{aligned}
G_1 &= \{x \mid x - 1 > 0 \wedge x + 2 > 0 \wedge x - 5 > 0\}\\
G_2 &= \{x \mid x - 1 > 0 \wedge x + 2 < 0 \wedge x - 5 < 0\}\\
G_3 &= \{x \mid x - 1 < 0 \wedge x + 2 > 0 \wedge x - 5 < 0\}\\
G_4 &= \{x \mid x - 1 < 0 \wedge x + 2 < 0 \wedge x - 5 > 0\}
\end{aligned}$$

Case 1: $G_1 = \{x \mid x - 1 > 0 \wedge x + 2 > 0 \wedge x - 5 > 0\}$
$= \{x \mid x > 1 \wedge x > -2 \wedge x > 5\}$
$= \{x \mid x > 5\}$

Case 2: $G_2 = \{x \mid x - 1 > 0 \wedge x + 2 < 0 \wedge x - 5 < 0\}$
$= \{x \mid x > 1 \wedge x < -2 \wedge x < 5\}$
$= \emptyset$

Case 3: $G_3 = \{x \mid x - 1 < 0 \wedge x + 2 > 0 \wedge x - 5 < 0\}$
$= \{x \mid x < 1 \wedge x > -2 \wedge x < 5\}$
$= \{x \mid x \in]-2,1[\}$

Case 4: $G_4 = \{x \mid x - 1 < 0 \wedge x + 2 < 0 \wedge x - 5 > 0\}$
$= \{x \mid x < 1 \wedge x < -2 \wedge x > 5\}$
$= \emptyset$

Hence $G = \{x \mid x > 5\} \cup \emptyset \cup \{x \mid x \in]-2,1[\} \cup \emptyset$
$= \{x \mid x \in]-2,1[\text{ or } x \in]5, \infty[\}$

which, interpreted graphically, is shown in Fig. 43.

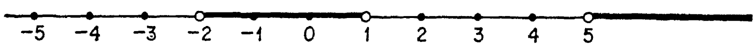


FIG. 43

Use the procedure of the example to interpret graphically each of the following sets:

a. $H = \{x \in R_e \mid x(x + 5)(x - 2) > 0\}$
b. $J = \{x \in R_e \mid (x - 2)(x - 4)(x + 2) < 0\}$
c. $K = \{x \in R_e \mid (x + 5)(x - 2)(x + 1) \leq 0\}$
d. $M = \{x \in R_e \mid (x + 2)(x - 3)^2 \geq 0\}$

3. To say that $\sqrt{2}$ is irrational means that no rational number exists whose square is equal to 2. The method of proof is attributed to Euclid and is referred to as "*reductio ad absurdum*" or "indirect proof." Either $\sqrt{2}$ is rational or it is irrational. The method of proof assumes that $\sqrt{2}$ is rational, and, as a consequence, various contradictions arise which eventually disprove this fact.

If $\sqrt{2}$ is rational, then there exists a rational number of the form p/q (where $p,q \in I$, $q \neq 0$) whose square is 2. It is understood that the rational number p/q is reduced to lowest terms; that is, p and q have no integral divisors other than unity. Hence,

Proof:	*Comments*
(1) $(p/q)^2 = 2$.	(1) Basic assumption.
(2) $p^2 = 2q^2$.	(2) Definition of division and laws of exponents from algebra.
(3) Since $p^2 = 2q^2$, then p^2 must be an even integer.	(3) q^2 is an integer and $2q^2$ is an even integer.
(4) If p^2 is even, then p must be even. Thus, p is divisible by 2 and p takes the form $2k$, where $k \in I$.	(4) If x is an even integer, then x^2 is an even integer and the converse is also true.
(5) In $p^2 = 2q^2$ we replace p by $2k$. Thus, $(2k)^2 = 2q^2$ and $4k^2 = 2q^2$.	(5) Substitution.

(6) If $4k^2 = 2q^2$, then $2k^2 = q^2$.	(6) Dividing through by 2.
(7) Since $q^2 = 2k^2$, then q^2 must be an even integer. Hence, q is also an even integer and is divisible by 2.	(7) Argument analogous to steps 3 and 4.
(8) The initial assumption that p/q was in lowest terms is contradictory to steps 4 and 7, which show that p and q have a common divisor 2.	(8) Contradiction.
(9) The initial assumption is incorrect, and the conclusion is that no rational number exists whose square is 2.	(9) Hence the square root of 2, designated as $\sqrt{2}$, is an irrational number.

a. Prove that no rational number exists whose square is 3.
b. Prove that no rational number exists whose cube is 3.

4. If a red die and a white die are to be rolled, where the possible outcomes are designated as the ordered pairs (x,y) with x representing the number on the red die and y the number on the white die, describe each of the subsets in parts *a* to *e* by the tabulation method and by the defining-property method.

Example. If we let $U = \{1,2,3,4,5,6\}$, then the set of all possible outcomes can be expressed as the cartesian product

$$\begin{aligned} U \times U = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),\\ &(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),\\ &(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

The subset S_1, where the dots on the two dice add to 7, is given by

$$\begin{aligned} S_1 &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\ &= \{(x,y) \mid x + y = 7\} \end{aligned}$$

The subset S_2, where the dots on the two dice add to 11, is given by

$$\begin{aligned} S_2 &= \{(5,6),(6,5)\} \\ &= \{(x,y) \mid x + y = 11\} \end{aligned}$$

The subset S_3, where the dots on the two dice add to 2, is given by

$$\begin{aligned} S_3 &= \{(1,1)\} \\ &= \{(x,y) \mid x + y = 2\} \end{aligned}$$

The subset S_4, where the dots on the two dice add to 13, is given by

$$\begin{aligned} S_4 &= \{\quad\} = \emptyset \\ &= \{(x,y) \mid x + y = 13\} \end{aligned}$$

a. The sum of the dots on the two dice is 3, 5, 8, 9, or 12.
b. The sum of the dots on the two dice is either 7 or 11.
c. The sum of the dots on the two dice is 2, 3, or 12.
d. The number of dots on the first die is three greater than the number on the second die.
e. The number of dots on the first die is twice the number on the second die.

5. When a nickel, a dime, and a quarter are tossed simultaneously, the universal set $U \times U \times U$ will contain eight elements that correspond to all the possible outcomes.

a. Tabulate the elements of $U \times U \times U$.

b. Let A = set of elements corresponding to the nickel falling tails
B = set of elements in which the three coins match
C = set of elements in which the number of heads exceeds the number of tails

Tabulate:

(1) A	(2) B	(3) $A \cup B$
(4) $A' \cup C$	(5) $B \cup C$	(6) $B' \cap C$
(7) $A' \cap C$	(8) $(A \cap B) \cap C'$	(9) $(A \cup C)'$

Hint: Representing nickels, dimes, and quarters with n, d, and q, respectively, and using a subscript h or t to represent heads or tails, we have

$$U \times U \times U = \{(n_h,d_h,q_h),(n_h,d_h,q_t),(n_h,d_t,q_h),(n_h,d_t,q_t),(n_t,d_h,q_h),(n_t,d_h,q_t),(n_t,d_t,q_h),(n_t,d_t,q_t)\}$$

6. Let $A = \{1,2,3,4\}$, $B = \{3,4,5\}$, $C = \{4,5,6,7\}$, and $D = \{3,4\}$. Tabulate each of the following:

a. $A \times A$	*b.* $A \times B$	*c.* $B \times B$
d. $B \times A$	*e.* $A \times C$	*f.* $B \times C$
g. $A \times D$	*h.* $B \times D$	*i.* $C \times D$
j. $(B \times C) \cap (A \times D)$	*k.* $(B \times D) \cap (B \times A)$	*l.* $A \times B \times D$

7. Let A, B, C, and D be the sets described in Problem 6.

a. Is $A \times B \times D$ equal to $B \times A \times D$? Is $n(A \times B \times D) = n(B \times A \times D)$?

b. Is $A \times B \times D$ equal to $B \times D \times A$?

c. Is $A \times B \times B$ equal to $B \times B \times A$?

d. Is $A \times A \times A$ equal to $D \times D \times D$?

e. Is $D \times D \times D$ a subset of $A \times A \times A$?

f. Is $A \times B \times D$ a subset of $A \times C \times D$?

8. If A, B, and C are the sets described in Problem 6, show that the following statements are true:

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

b. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

9*a.* The cartesian set $A \times A$ has 16 ordered pairs. If two of these members are (3,1) and (4,2), find the remaining 14 members. Tabulate the elements of set A.

b. The cartesian set $A \times A \times A$ has 27 ordered triples, and one of its elements is (0,1,2). Find the remaining 26 elements.

10. A rational number is defined to be a number of the form p/q, where $p,q \in I$ and $q \neq 0$. Another definition for p/q is obtained from the ordered pair (p,q). For example, the rational numbers $\frac{3}{4}$ and $-\frac{5}{8}$ can be represented as the ordered pairs (3,4) and (−5,8), respectively. Equality and the operations of addition and multiplication for rational numbers when expressed as ordered pairs of integers are now defined.

If $(a,b) \in F$, $(c,d) \in F$, $b \neq 0$, and $d \neq 0$, then:

a. $(a,b) = (c,d)$ if and only if $ad = bc$.

b. $(a,b) + (c,d) = (ad + bc, bd)$.

c. $(a,c) \cdot (c,b) = (ac,bd)$.

Example.

a. The ordered pairs (3,4) and (6,8) are equal, since $3 \cdot 8 = 4 \cdot 6$. In the form p/q, $\frac{3}{4} = \frac{6}{8}$ and $3 \cdot 8 = 6 \cdot 4$.

b. $(7,8) + (3,5) = (35 + 24, 40)$
$= (59,40)$

c. $(5,6) \cdot (3,10) = (15,60)$
$= (1,4)$ since $(15,60) = (1,4)$

If rational numbers are interpreted as ordered pairs of integers, whose properties are accepted, then the laws for the system of rational numbers (Section 2.5) can be expressed as follows.

If $(a,b),(c,d),(e,f) \in F$, where $a,b,c,d,e,f \in I$ and $b \cdot d \cdot f \neq 0$, then:

Closure Laws

F-1: $(a,b) + (c,d) = (ad + bc, bd)$ and $(ad + bc, bd) \in F$.
Example. $(3,4) + (2,3) = (9 + 8, 12) = (17,12)$, and $(17,12) \in F$.

F-2: $(a,b) \cdot (c,d) = (ac,bd)$ and $(ac,bd) \in F$.
Example. $(3,4) \cdot (5,11) = (15,44)$, and $(15,44) \in F$.

Commutative Laws

F-3: $(a,b) + (c,d) = (c,d) + (a,b)$.

Verification:

	Authority
$(a,b) + (c,d) = (ad + bc, bd)$	Definition of addition
$(c,d) + (a,b) = (cb + da, db)$	Definition of addition
$= (ad + bc, bd)$	Addition and multiplication of integers are commutative

Hence $(a,b) + (c,d) = (c,d) + (a,b)$.

F-4: $(a,b) \cdot (c,d) = (c,d) \cdot (a,b)$.

Associative Laws

F-5: $[(a,b) + (c,d)] + (e,f) = (a,b) + [(c,d) + (e,f)]$.

F-6: $[(a,b) \cdot (c,d)] \cdot (e,f) = (a,b) \cdot [(c,d) \cdot (e,f)]$.

Distributive Law

F-7: $(a,b) \cdot [(c,d) + (e,f)] = (a,b) \cdot (c,d) + (a,b) \cdot (e,f)$.

Verification:

	Authority
$(a,b) \cdot [(c,d) + (e,f)] = (a,b) \cdot (cf + de, df)$	Definition of addition and distributive law for integers
$= (acf + ade, bdf)$	Definition of multiplication
$(a,b) \cdot (c,d) + (a,b) \cdot (e,f)$	
$= (ac,bd) + (ae,bf)$	Definition of multiplication
$= (acbf + abde, bbdf)$	Definition of addition
$= (acf + ade, bdf)$	Definition of equality

Hence $(a,b) \cdot [(c,d) + (e,f)] = (a,b) \cdot (c,d) + (a,b) \cdot (e,f)$.

Identity Laws

F-8: There exists a unique element (1,1) in F, such that $(a,b) \cdot (1,1) = (a,b)$.

F-9: There exists a unique element (0,1) in F, such that $(a,b) + (0,1) = (a,b)$.

Inverse Elements

F-10: For every element (a,b) of F there exists a unique element $(-a,b)$ of F, such that $(a,b) + (-a,b) = (0,1)$.

Verification:	*Authority*
$(a,b) + (-a,b) = (ab - ab,\ b)$	Definition of addition
$= (0,b)$	Additive inverse for integers
$= (0,1)$	Definition of equality

Hence $(a,b) + (-a,b) = (0,1)$.

F-13: For every element (a,b) of F, except $(0,1)$, there exists a unique element (b,a), such that $(a,b) \cdot (b,a) = (1,1)$.

Verification:	*Authority*
$(a,b) \cdot (b,a) = (ab,ba)$	Definition of multiplication
$= (ab,ab)$	Commutative property of integers
$= (1,1)$	Definition of equality

Hence $(a,b) \cdot (b,a) = (1,1)$.

The interpretation of Laws F-11 and F-12 is left as an exercise.

Properties of Equality

Let $(a,b),(c,d),(e,f) \in F$.

E-1: $(a,b) = (a,b)$.
E-2: If $(a,b) = (c,d)$, then $(c,d) = (a,b)$.
E-3: If $(a,b) = (c,d)$ and $(c,d) = (e,f)$, then $(a,b) = (e,f)$.
E-4: If $(a,b) = (c,d)$, then $(a,b) + (e,f) = (c,d) + (e,f)$.
E-5: If $(a,b) = (c,d)$, then $(a,b) \cdot (e,f) = (c,d) \cdot (e,f)$.

We leave the verification of these laws and properties of rational numbers as an exercise. Each law should be tested with particular ordered pairs, so as to gain an intuitive acceptance for the property. It is not necessarily advocated that rational numbers be used in the form of ordered pairs for computation. This concept has been introduced to provide a deeper insight into the structuring of numbers, for example, the extension made possible by considering rational numbers as ordered pairs of integers.

11. In a manner similar to that discussed in Problem 10, the system of integers may be obtained as an extension of the system of natural numbers. Every integer may be expressed as an ordered pair of natural numbers through the following scheme.

The integer 3 may be expressed as the difference of two natural numbers such as $5 - 2$ or $7 - 4$. Similarly, 0 may be represented as the difference $5 - 5$ or $7 - 7$. Thus if an integer x is defined to be the difference $a - b$, where a and b are natural numbers, then x may be represented by the ordered pair (a,b). Many ordered pairs of natural numbers may be utilized to represent a particular integer. For example, the integer -5 can be represented by the ordered pairs $(1,6),(4,9),(2,7)$, and many others. Equality between two ordered pairs of natural numbers $(a,b) = (c,d)$ is defined to mean that $a + d = b + c$. Hence $(2,8) = (5,11)$, since $2 + 11 = 8 + 5$, and $(1,5) = (3,7)$, since $1 + 7 = 5 + 3$.

The operations of addition and multiplication are defined as follows:

$$(a,b) + (c,d) = (a + c,\ b + d)$$
$$(a,b) \cdot (c,d) = (ac + bd,\ ad + bc)$$

Each of the laws for the system of integers may be stated in terms of the concept of an ordered pair. These laws can be verified by use of the definitions for equality, addition, and multiplication of ordered pairs of natural numbers.

a. Write and verify the laws for the system of integers using the ordered-pair notation. A few suggestions that may be helpful are:

(1) The identity element for addition is (1,1).

Verification:	*Authority*
$(a,b) + (1,1) = (a + 1, b + 1)$	Definition of addition
$= (a,b)$	Definition of equality

(2) The identity element for multiplication is $(a + 1, a)$.

(3) The additive inverse of (a,b) is (b,a).

Verification:	*Authority*
$(a,b) + (b,a) = (a + b, a + b)$	Definition of addition and commutative law for natural numbers
$= (1,1)$	Definition of equality

b. A positive integer may be expressed as $(a + b, a)$, while a negative integer may be written $(c, c + d)$, where a, b, c, and d are natural numbers. To prove that the product of two negative integers is a positive integer, it is necessary to show that the product of two integers of the form $(g, g + h)$ and $(k, k + l)$ is an integer of the form $(m + n, m)$ where $g,h,k,l,m,n \in N$. This is left as an exercise.

c. Prove that the product of a negative integer and a positive integer is a negative integer.

4

Relations and Functions

4.1 INTRODUCTION

An important consequence of the language of sets has been the evolvement of a more precise meaning for the concepts of "relation" and "function." These notions must not be confused with their everyday meanings but are to be accepted only in accord with their mathematical definitions.

The successful student in mathematics needs manipulative skill in formal courses such as algebra, trigonometry, analytics, and the like, but the mere attainment of such proficiency without understanding is certainly undesirable. A greater appreciation for the unity of elementary mathematics is provided by the fundamental ideas of relation and function.

4.2 SUBSETS OF $R_e \times R_e$

In previous sections, subsets of R_e were described by means of equations and inequalities. For example, $\{x \in R_e \mid x > 3\}$ was interpreted geometrically as a part of a one-dimensional space, the real-number line. In general, a set such as $\{x \in R_e \mid P_x\}$, where P_x represents the defining condition involving the variable x, contains real numbers as elements where each real number corresponds to one and only one point on the real-number line.

Accordingly, for $R_e \times R_e = \{(x,y) \mid x \in R_e \wedge y \in R_e\}$, a 1-1 correspondence is established between the set of ordered pairs belonging to $R_e \times R_e$ and the set of points in a plane. It follows that the graphical interpretation of $R_e \times R_e$ is the entire plane. In the same manner as subsets of R_e were studied through the use of equations and inequalities, similar media may be used to describe subsets of $R_e \times R_e$. These subsets of ordered pairs are known by another name, relations, and correspond to subspaces of the xy plane. Hence, relations in $R_e \times R_e$

take the form $\{(x,y) \mid x \in R_e \wedge y \in R_e \wedge P_{xy}\}$ or $\{(x,y) \in R_e \times R_e \mid P_{xy}\}$, where P_{xy} is the defining condition involving the variables x and y. The examples which follow give further emphasis to the ways of describing relations in $R_e \times R_e$. As is customary, relations will be designated by capital letters, and when no universe is specified, it is understood to be $R_e \times R_e$.

Example 1. The graph of $B = \{(x,y) \mid x \in]-\infty,\infty[\text{ and } y \in]-\infty,\infty[\}$ is the entire xy plane as indicated in Fig. 44.
Example 1 illustrates another way of defining the set $R_e \times R_e$.

Example 2. Line L of Fig. 45 can be described as

$$\begin{aligned} S &= \text{the infinitude of points on line } L \\ &= \{(x,y) \mid y = x\} \\ &= \{(-3,-3),(0,0),(1,1),(\sqrt{2},\sqrt{2}),(1.7,1.7),(4,4), \ldots\} \end{aligned}$$

Example 3. If relation A is described as

$$\begin{aligned} A &= \{(x,y) \mid x = 2\} \\ &= \{(2,3),(2,-1),(2,0),(2,-3), \ldots\} \end{aligned}$$

its graph is represented in Fig. 46.

It is noted that there is a distinct difference between

$$A = \{(x,y) \in R_e \times R_e \mid x = 2\} = \{(2,3),(2,-1),(2,0), \ldots\}$$

and $B = \{x \mid x \in R_e \text{ and } x = 2\} = \{2\}$

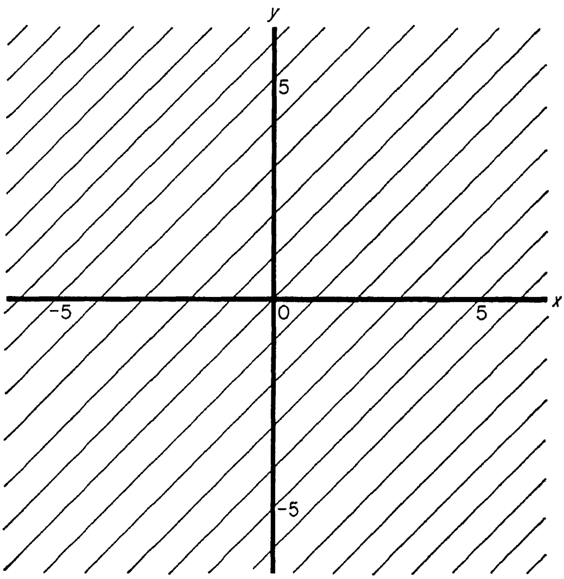


FIG. 44

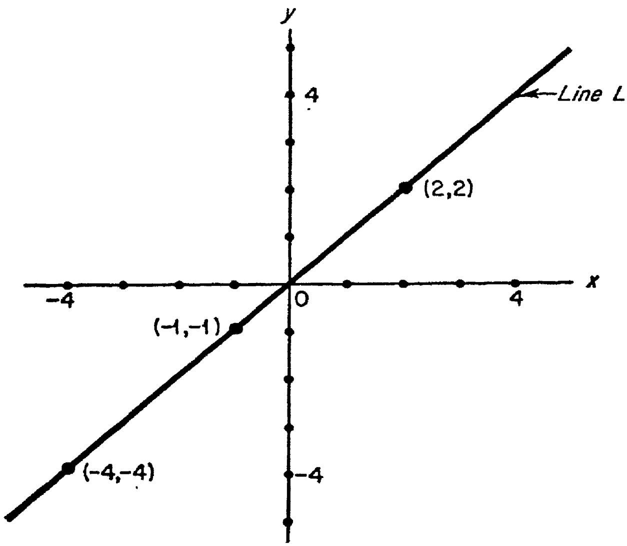


FIG. 45

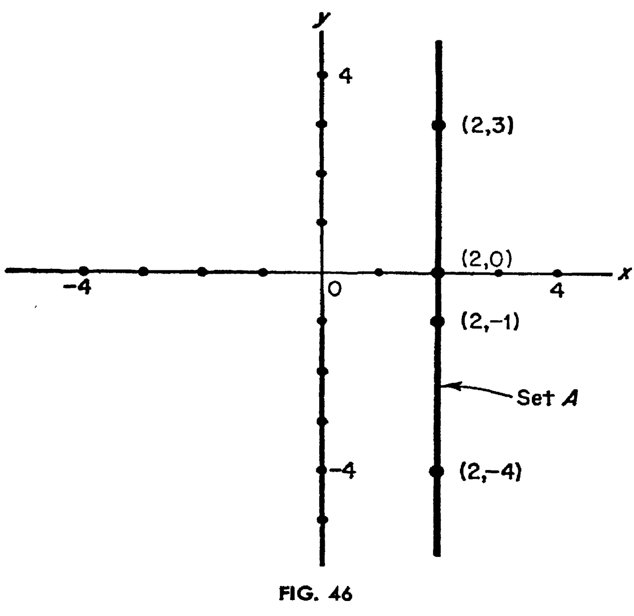


FIG. 46

The tabulation of set B exposes only the single real number 2, while the tabulation of set A exposes an infinitude of ordered pairs of real numbers. Thus, set B is interpreted graphically in a one-dimensional space (real-number line), while set A is interpreted graphically in a two-dimensional space (xy plane).

Example 4. The relation $C = \{(x,y) \mid x = 4 \text{ and } y \in [-2,4]\}$ is graphed in Fig. 47.

Note that the defining condition $x = 4$ can be written $x + 0 \cdot y = 4$.

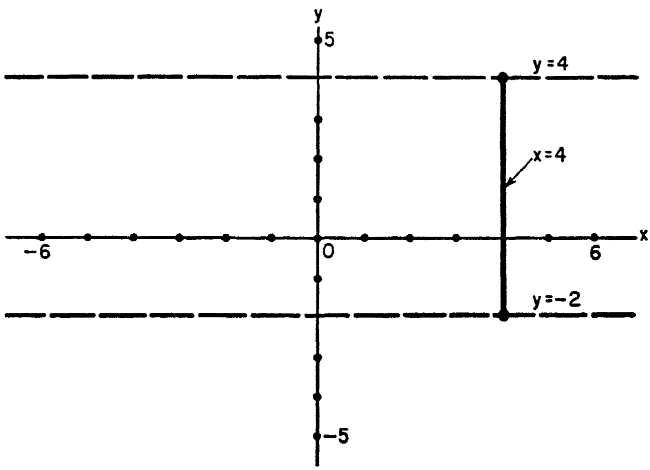


FIG. 47

Example 5a. The set of points on the x axis can be described as

$$\begin{aligned} X &= \text{the infinitude of points on the } x \text{ axis} \\ &= \{(x,y) \in R_e \times R_e \mid y = 0\} \end{aligned}$$

b. The set of points on the y axis can be described as

$$\begin{aligned} Y &= \text{the infinitude of points on the } y \text{ axis} \\ &= \{(x,y) \in R_e \times R_e \mid x = 0\} \end{aligned}$$

c. The upper portion of the y axis, excluding the origin, can be described as

$$\begin{aligned} Y^{+} &= \text{the infinitude of points on the upper part of the } y \text{ axis} \\ &= \{(x,y) \in R_e \times R_e \mid x = 0 \wedge y > 0\} \end{aligned}$$

Example 6. The relation $A = \{(x,y) \mid x \in [-2,4] \text{ and } y \in [0,3]\}$ is graphed in Fig. 48 and is represented by the shaded portion.

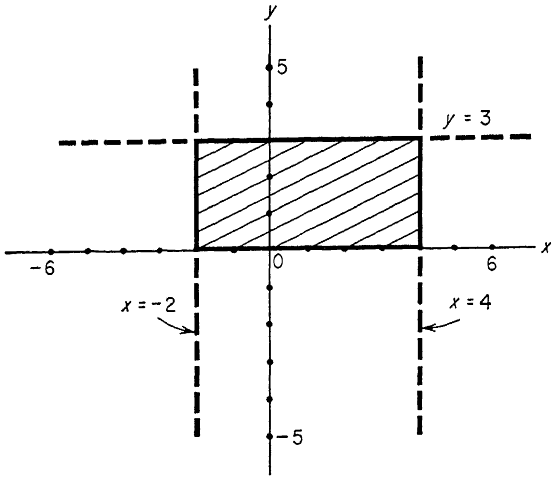


FIG. 48

Example 7. The following examples illustrate relations which are subsets of other relations. If $A = \{(x,y) \mid x^2 + y^2 = 25\}$, where x and y represent the coordinates of the points on the circle, then the graphs of relations B, C, D, and E, which are subsets of A, appear in Figs. 49 to 52.

a. $B = \{(x,y) \mid (x,y) \in A \text{ and } y = 0\}$
$\quad = \{(-5,0),(5,0)\}$

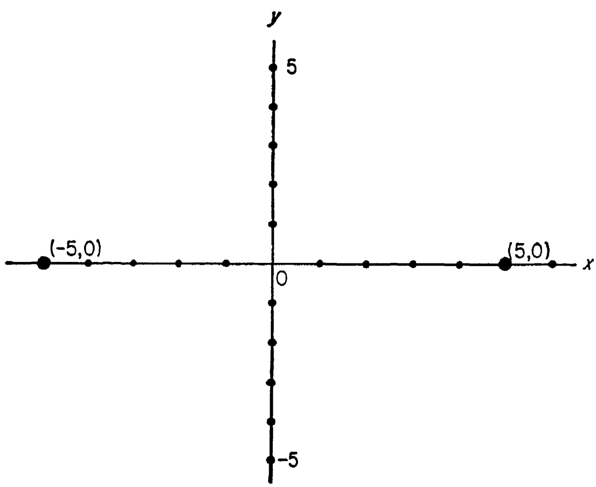


FIG. 49

b. $C = \{(x,y) \mid (x,y) \in A \text{ and } y \geq 0\}$
$=$ the set of all points on the circle on and above the x axis

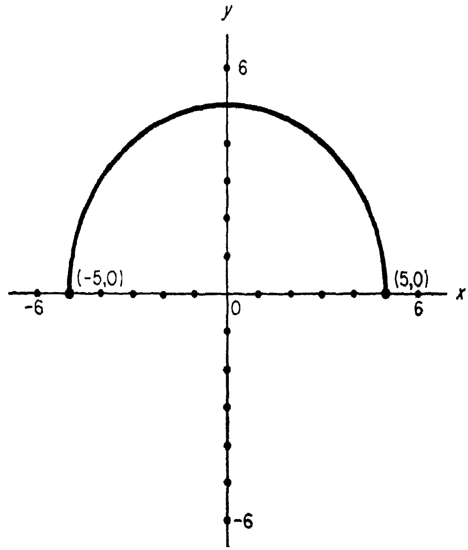


FIG. 50

c. $D = \{(x,y) \mid (x,y) \in A \wedge x > 0\}$
$=$ the set of all points on the circle to the right of the y axis, excluding $(0,5)$ and $(0,-5)$

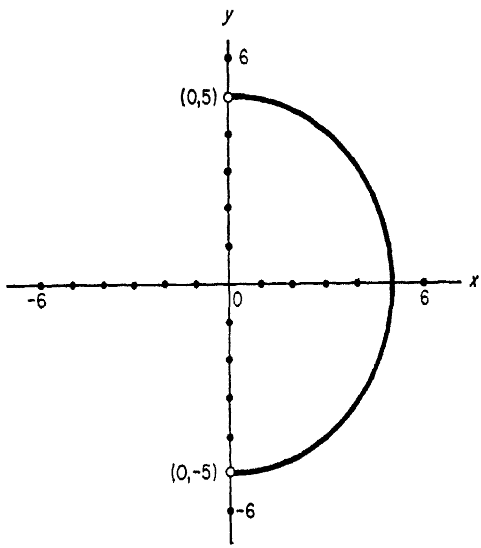


FIG. 51

d. $E = \{(x,y) \mid (x,y) \in A \wedge y = -\sqrt{25 - x^2}\}$
$=$ the set of all points on the circle below the x axis, including $(5,0)$ and $(-5,0)$

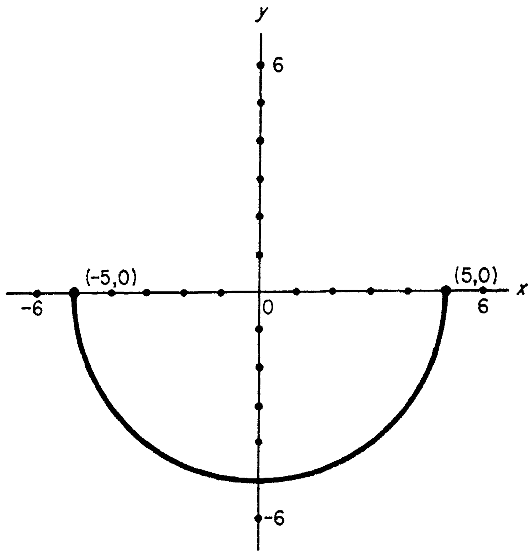


FIG. 52

4.3 INEQUALITIES IN TWO VARIABLES

Defining conditions which involve two variables x and y with the connectives "$=$," "$<$," and "$>$" as implied in $\alpha x + \beta y = \gamma$, $\alpha x + \beta y > \gamma$, and $\alpha x + \beta y < \gamma$ (where $\alpha,\beta,\gamma \in R_e$) produce relations of special interest. A line L is determined by the defining condition $\alpha x + \beta y = \gamma$. This line divides the xy plane into three distinct sets of points. Each ordered pair (x,y) corresponding to a point in the plane is an element belonging to one of three sets or relations, either S_1, S_2, or S_3, where

$$S_1 = \{(x,y) \mid \alpha x + \beta y = \gamma\}$$
$$S_2 = \{(x,y) \mid \alpha x + \beta y > \gamma\}$$
$$S_3 = \{(x,y) \mid \alpha x + \beta y < \gamma\}$$

For example, the defining condition $x + 2y = 4$ produces the line so labeled in Fig. 53. This in turn divides the plane into the three regions corresponding to the relations

$$S_1 = \{(x,y) \mid x + 2y = 4\}$$
$$S_2 = \{(x,y) \mid x + 2y > 4\}$$
$$S_3 = \{(x,y) \mid x + 2y < 4\}$$

For convenience, the corresponding subspaces of the plane are designated as regions S_1, S_2, and S_3. The defining conditions that describe S_2 and S_3 are given by the inequalities $x + 2y > 4$ and $x + 2y < 4$, respectively.

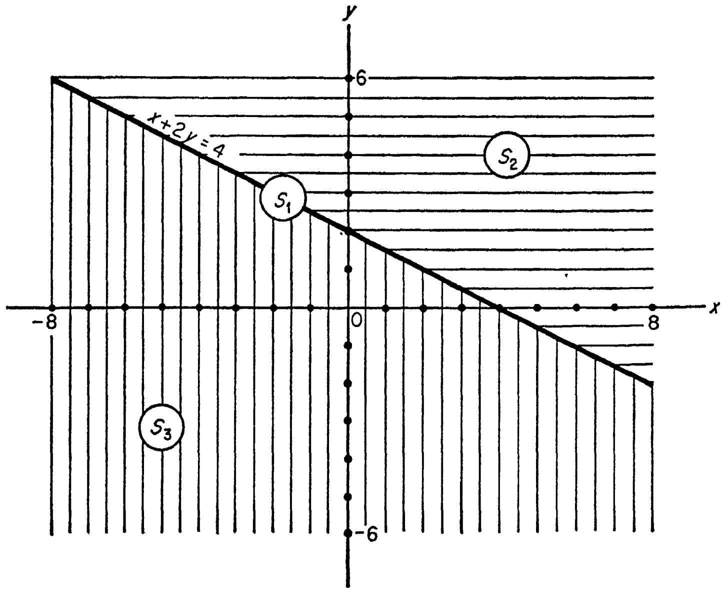


FIG. 53

The ordered pairs (0,0), (0,1), and (1,0) can be used effectively as testing pairs that distinguish these regions one from another. In this example, the use of the ordered pair (0,0) and the consequent substitution of zero for x and zero for y in $x + 2y > 4$ leads to $0 > 4$, which is a false statement. Accordingly, it will be found that all other points (x,y) on the same side of the line as (0,0) in region S_3 continue to yield false statements when substituted in $x + 2y > 4$. Since region S_1 includes only those points on the line, it follows that the region S_2 is described by the defining condition $x + 2y > 4$.

In more general terms, this procedure may be summarized as follows:

Step 1. Study the line $\alpha x + \beta y = \gamma$ suggested by either of the inequalities $\alpha x + \beta y > \gamma$ or $\alpha x + \beta y < \gamma$.

Step 2. Represent sets S_1, S_2, and S_3 graphically.

Step 3. The testing procedure should exclude those points associated with (0,0), (1,0), or (0,1) that are on the line obtained from $\alpha x + \beta y = \gamma$. There is always at least one such point that is not on this line.

Step 4. Substitute the x and y of any one of the points not on the line in the given inequality. If the substitution results in a true statement, all other points on the same side of the line as the testing point will continue to yield true statements and belong to the same set.

Step 5. The judgments made in Step 4 will uniquely distinguish S_1, S_2, and S_3 graphically and will properly associate them with the defining conditions $\alpha x + \beta y = \gamma$, $\alpha x + \beta y > \gamma$, and $\alpha x + \beta y < \gamma$.

Example 1. The lined region of Fig. 54 represents the relation

$$D = \{(x,y) \mid x + y \geq 3\}$$

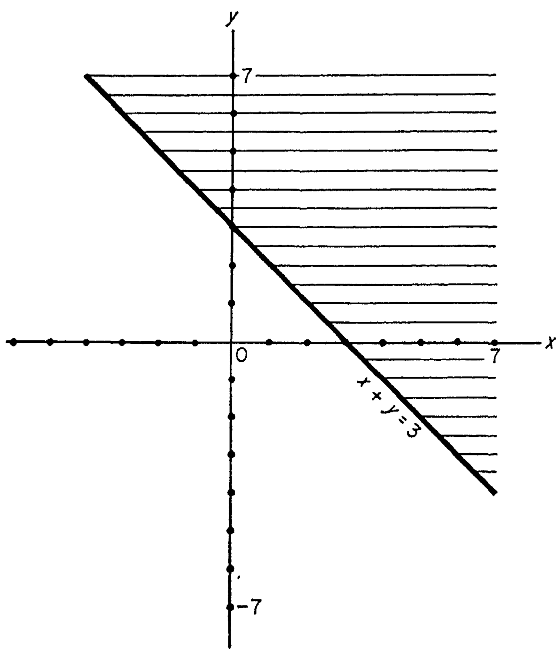


FIG. 54

In the event that the defining condition is given as $x + y > 3$, a dotted line replaces the heavy line labeled $x + y = 3$. This indicates that the points on the line are excluded.

Example 2. The graph of the set or relation $\{(x,y) \mid x^2 + y^2 = 25\}$, where $x \in R_e$ and $y \in R_e$, partitions the plane into the three regions S_1, S_2, and S_3. These regions or subspaces are shown in Fig. 55, where S_1 represents the points on the circle, S_3 those within the circle, and S_2

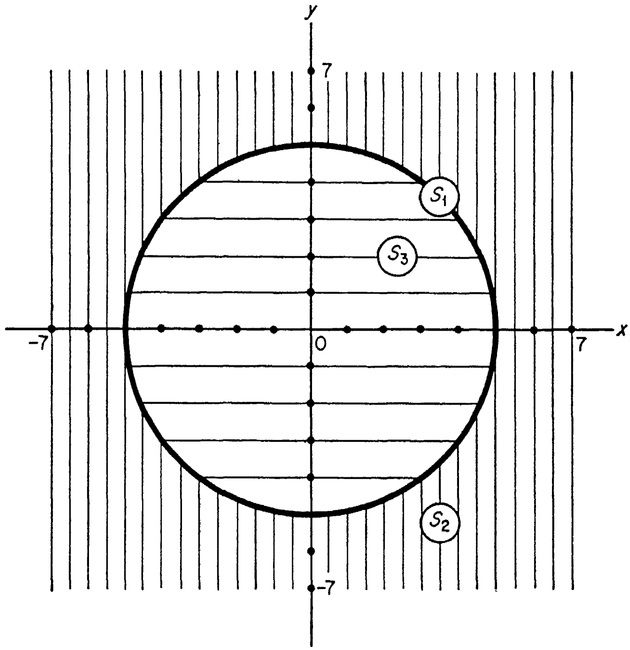


FIG. 55

those outside the circle. The set description of each of these regions is given in Table 1. Note that $S_1 \cup S_2 \cup S_3 = R_e \times R_e$.

Table 1

Region	Set description	Subspace description
S_1	$\{(x,y) \mid x^2 + y^2 = 25\}$	Curve (circle)
S_2	$\{(x,y) \mid x^2 + y^2 > 25\}$	Vertical shading
S_3	$\{(x,y) \mid x^2 + y^2 < 25\}$	Horizontal shading

To determine the specific region corresponding to a given relation, such as in Example 2, it is convenient to use the following procedure:

Step 1. Represent the three sets S_1, S_2, and S_3 graphically.

Step 2. Test representative points from each of the regions in the defining condition of the given relation. For example, $(0,0) \in S_3$. If x and y are each replaced by 0 in $x^2 + y^2 < 25$, the true statement $0 < 25$ is obtained. All other points of region S_3 possess coordinates whose replacement in $x^2 + y^2 < 25$ continue to yield true statements. In the event that the chosen point had resulted in a false statement, then all other points of this same region would continue to do likewise. In

Example 2, any point either on or outside the circle has coordinates whose replacement in $x^2 + y^2 < 25$ produces a false statement. Accordingly, the proper association of a relation with a definite region may be accomplished by testing the validity of the defining condition in terms of the coordinates of a few points belonging to this region.

Example 3. The parabola defined by the equation $y = x^2 - 2x - 3$ divides the plane into three distinct regions T_1, T_2, and T_3, as shown in Fig. 56. Set descriptions for these regions are given in Table 2.

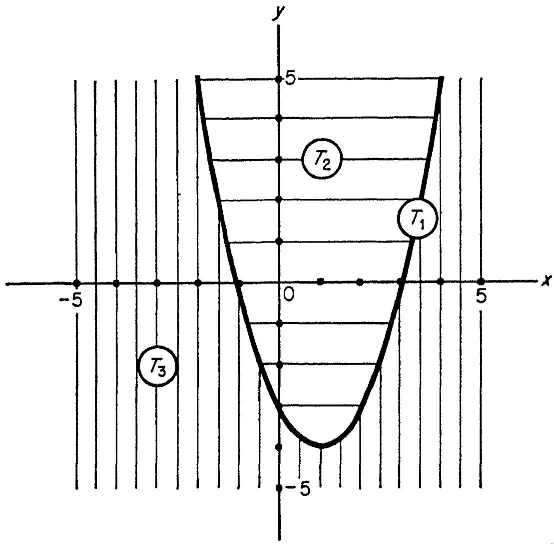


FIG. 56

Table 2

Region	Set description	Subspace description
T_1	$\{(x,y) \mid y = x^2 - 2x - 3\}$	Curve (parabola)
T_2	$\{(x,y) \mid y > x^2 - 2x - 3\}$	Horizontal shading
T_3	$\{(x,y) \mid y < x^2 - 2x - 3\}$	Vertical shading

Example 4. The ellipse given by the defining condition

$$9x^2 + 25y^2 = 225$$

divides the plane into the three distinct regions W_1, W_2, and W_3, as shown in Fig. 57. Table 3 gives set descriptions for these regions.

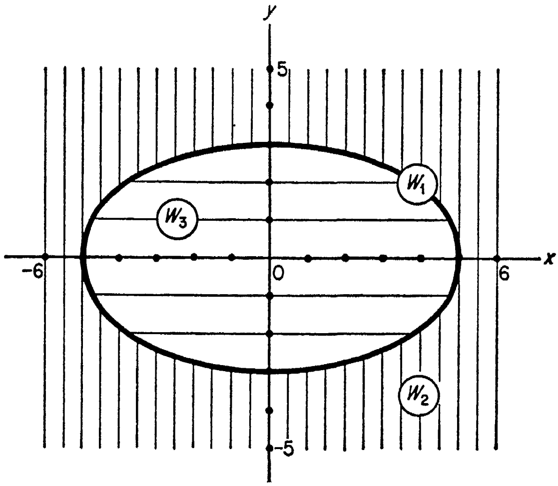


FIG. 57

Table 3

Region	Set description	Subspace description
W_1	$\{(x,y) \mid 9x^2 + 25y^2 = 225\}$	Curve (ellipse)
W_2	$\{(x,y) \mid 9x^2 + 25y^2 > 225\}$	Vertical shading
W_3	$\{(x,y) \mid 9x^2 + 25y^2 < 225\}$	Horizontal shading

Example 5. The hyperbola defined by the equation $9x^2 - 25y^2 = 225$ partitions the plane into the three distinct regions V_1, V_2, and V_3, as shown in Fig. 58. Set descriptions for these regions are given in Table 4.

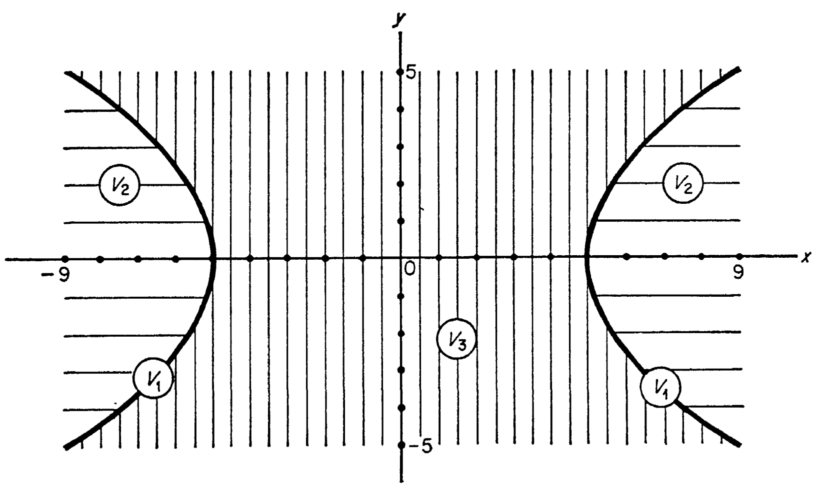


FIG. 58

Table 4

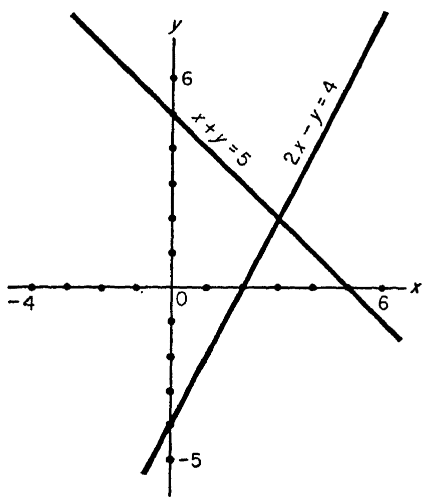
Region	Set description	Subspace description
V_1	$\{(x,y) \mid 9x^2 - 25y^2 = 225\}$	Curve (hyperbola)
V_2	$\{(x,y) \mid 9x^2 - 25y^2 > 225\}$	Horizontal shading
V_3	$\{(x,y) \mid 9x^2 - 25y^2 < 225\}$	Vertical shading

4.4 RELATIONS INVOLVING COMPOUND CONDITIONS

In relations, defining conditions may be connected by symbols "$\vee$" and "$\wedge$," read as "disjunction" and "conjunction" symbols. These symbols are used in the "calculus of propositions" to join statements together, not sets. Previous discussion of these symbols referred to conditions where just one variable was involved, but here their use is extended to defining conditions involving at least two variables. The symbol "$\vee$" means "inclusive or," which implies "this or that or both"; while the symbol "$\wedge$" means "and," which implies "this and that at the same time." Occasionally, the symbol "$\underline{\vee}$" is employed to mean "exclusive or," which implies "this or that but not both." The examples which follow illustrate the use of these connectives.

Example 1. The set or relation

$$G = \{(x,y) \mid x + y = 5 \vee 2x - y = 4\}$$



Graph of set G

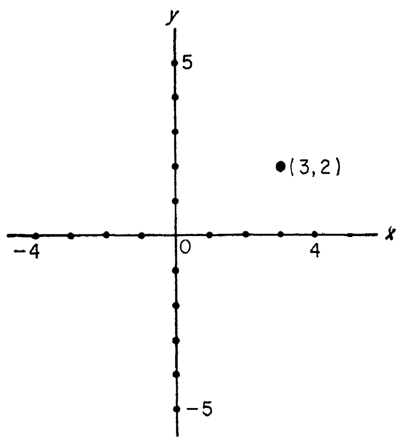
FIG. 59

means the totality of all ordered pairs $(x,y) \in R_e \times R_e$ which satisfy either $x + y = 5$ or $2x - y = 4$ or both. When translated graphically, G is represented by all the points on the two lines labeled with their respective defining conditions, as shown in Fig. 59.

Example 2. The set or relation

$$H = \{(x,y) \mid x + y = 5 \wedge 2x - y = 4\}$$

includes only those ordered pairs that correspond graphically to the points which are common to both lines, namely, the point of intersection. The relation $H = \{(3,2)\}$ is graphically represented by the point labeled (3,2) in Fig. 60.



Graph of set H

FIG. 60

In an algebra text, Example 1—

$$G = \{(x,y) \mid x + y = 5 \vee 2x - y = 4\}$$

—might have been stated as follows. Graph $x + y = 5$ and $2x - y = 4$ with respect to the same set of rectangular axes.

Example 2—$H = \{(x,y) \mid x + y = 5 \wedge 2x - y = 4\}$—might have been stated as follows. Solve simultaneously

$$\begin{Bmatrix} 2x - y = 4 \\ x + y = 5 \end{Bmatrix}$$

This is usually referred to as an "independent linear system."

It is important to observe that sets G and H could have been described as follows:

$$G = \{(x,y) \mid 2x - y = 4\} \cup \{(x,y) \mid x + y = 5\}$$
$$H = \{(x,y) \mid 2x - y = 4\} \cap \{(x,y) \mid x + y = 5\}$$

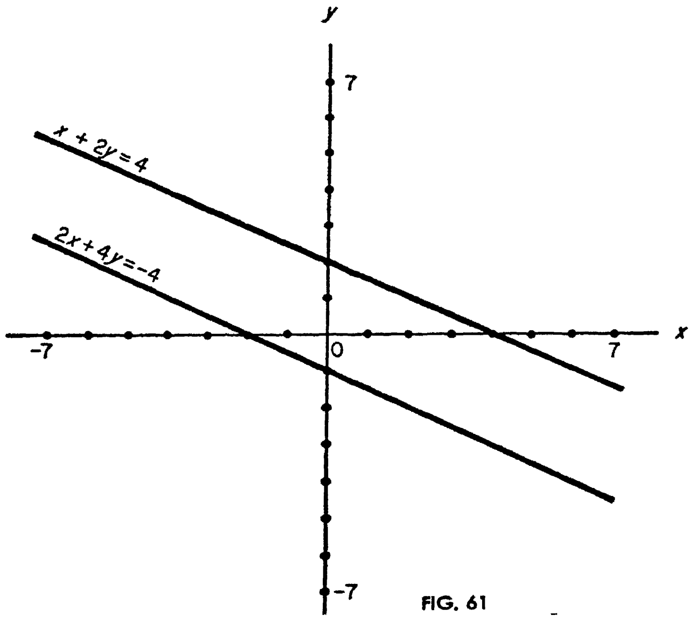


FIG. 61

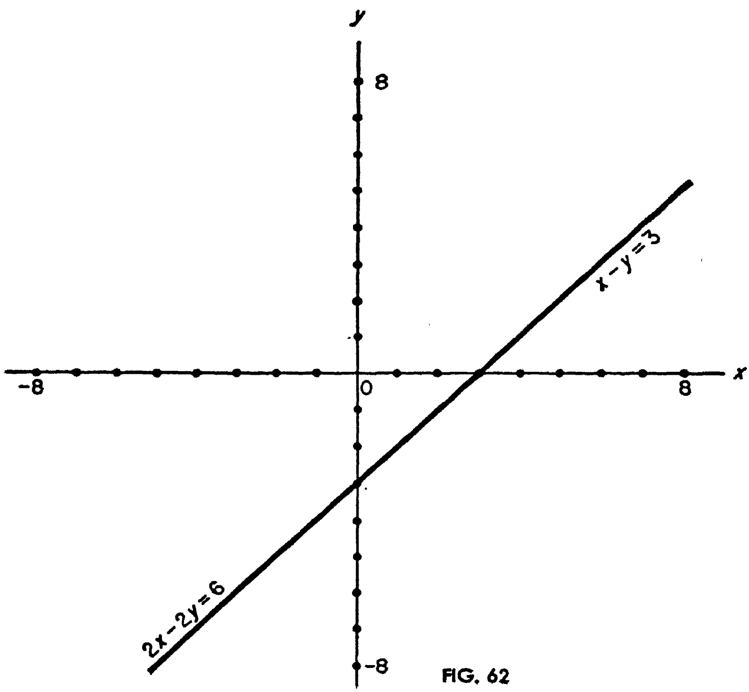


FIG. 62

Example 3. If $K = \{(x,y) \mid x + 2y = 4 \wedge 2x + 4y = -4\}$, then this results in what is referred to as an "inconsistent linear system," the graph of which is two distinct lines that are parallel, as shown in Fig. 61. Here $K = \{(x,y) \mid x + 2y = 4 \wedge 2x + 4y = -4\} = \emptyset$; that is, no ordered pairs exist that simultaneously satisfy both defining conditions.

Example 4. If

$$A = \{(x,y) \mid x - y = 3\}$$

and

$$B = \{(x,y) \mid 2x - 2y = 6\}$$

then $A \cap B = A$ or $A \cap B = B$ since the graph of A coincides with that of B as shown in Fig. 62. Since $x - y = 3$ implies $2x - 2y = 6$, we have equivalent defining conditions; that is, the solution set of $x - y = 3$ is identical to that of $2x - 2y = 6$. Such a system is referred to as a "dependent linear system."

Example 5. The lined region of Fig. 63 represents the relation

$$B = \{(x,y) \mid x \geq 4 \wedge y \geq 3\}$$

Compare with Example 6.

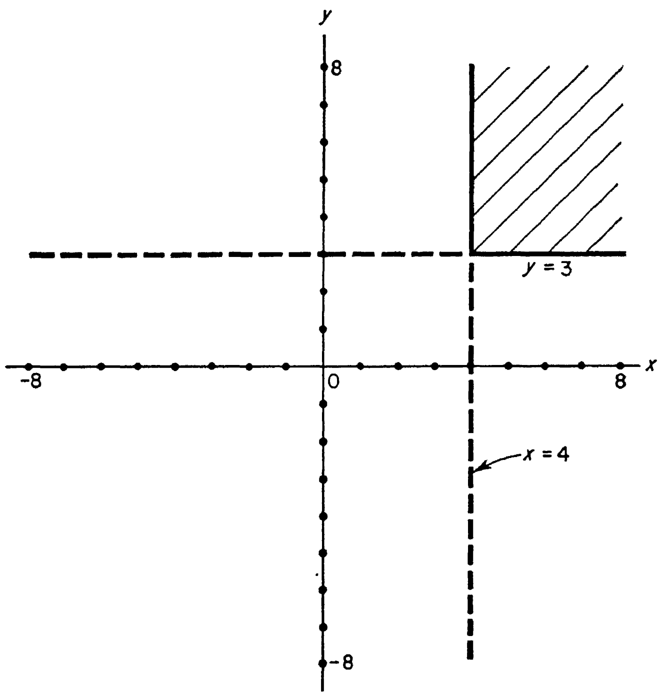


FIG. 63

Example 6. The lined regions of Fig. 64 represent the relation

$$C = \{(x,y) \mid |x| \geq 3 \wedge |y| \geq 2\}$$

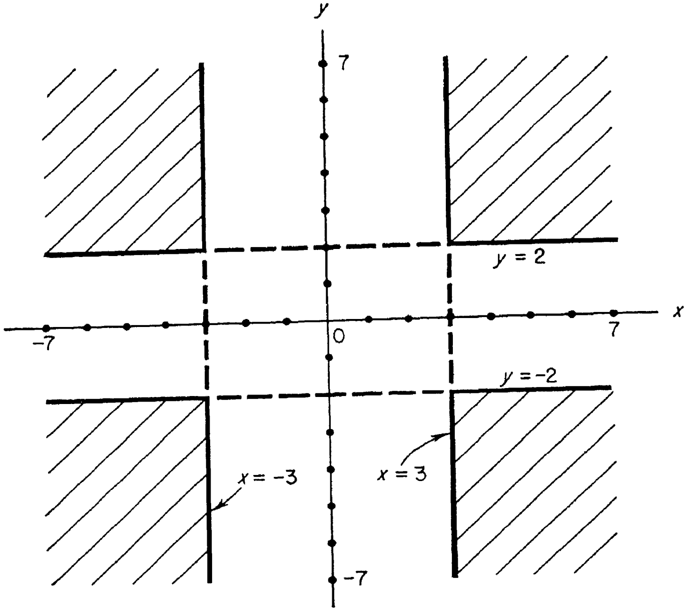


FIG. 64

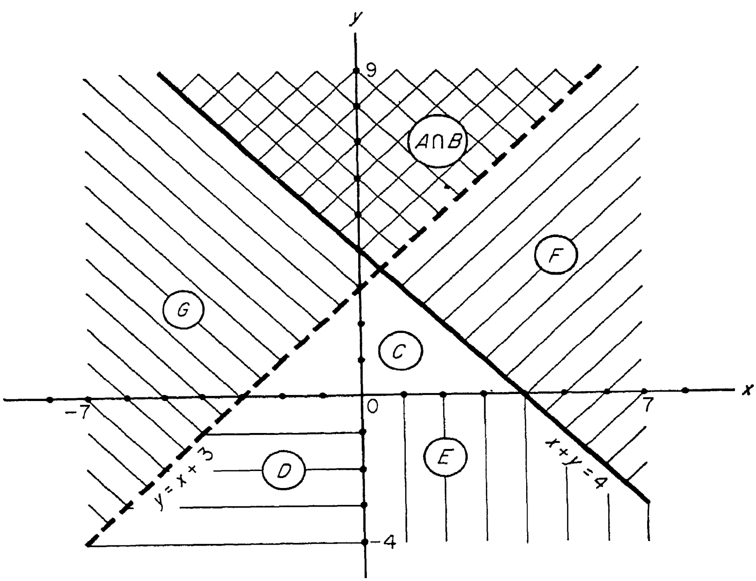


FIG. 65

Example 7. If $A = \{(x,y) \mid y > x + 3\}$ and $B = \{(x,y) \mid x + y \geq 4\}$, then $A \cap B$ is represented by the cross-hatched region shown in Fig. 65. The following set descriptions correspond to the indicated regions of Fig. 65:

$$C = \{(x,y) \mid x + y \leq 4 \wedge y < x + 3 \wedge y \geq 0\}$$
$$D = \{(x,y) \mid y < x + 3 \wedge x \leq 0 \wedge y \leq 0\}$$
$$E = \{(x,y) \mid x + y \leq 4 \wedge y \leq 0 \wedge x \geq 0\}$$
$$F = \{(x,y) \mid x + y \geq 4 \wedge y < x + 3\}$$
$$G = \{(x,y) \mid x + y \leq 4 \wedge y > x + 3\}$$

The inclusion of the boundary line in the description of any of the regions C, D, E, F, and G is largely a matter of choice. In the absence of information to the contrary we have included the boundary line in sets E and F. If directions had been given to describe disjoint sets, then we could have included the boundary line in either E or F but not both or could have considered each of the three sets E, F, and the boundary line separately.

If $A = \{(x,y) \mid y > x + 3\}$, then $A' = \{(x,y) \mid y \leq x + 3\}$ and $A^{-1} = \{(x,y) \mid x > y + 3\}$. We leave as an exercise the description and graphical interpretation of the following: $A \cup B$, B', $A' \cup B'$, $A' \cap B'$, $A \cup A'$, $A \cap A'$, B^{-1}, $A \cup A^{-1}$, $B \cup B^{-1}$, and $A \cap A^{-1}$.

Example 8. If $E = \{(x,y) \mid x + y \leq 2 \wedge x^2 + y^2 \leq 25\}$, then its graphical interpretation is given in Fig. 66. The relation E is represented

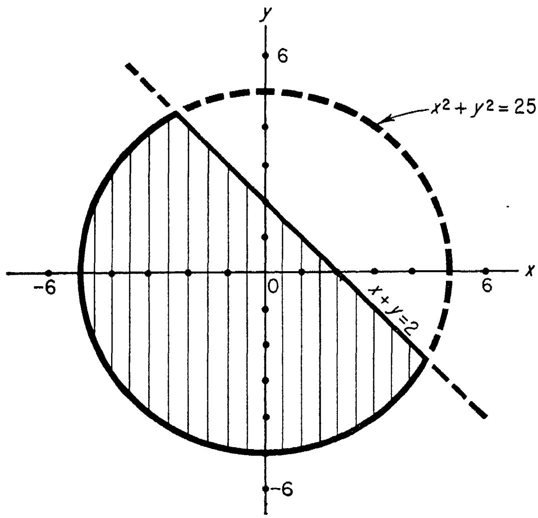


FIG. 66

by the lined area, since the coordinates of every point in this region simultaneously satisfy both the linear and the quadratic inequality.

Example 9. We include this example because it illustrates the use of relations where the defining conditions involve both linear and quadratic equations and inequalities in two variables. Further, it provides a mathematical environment in which artistic talents can be employed. Here we graph each of the indicated sets of rows a through l (in Table 1) on one set of axes, color each of the regions according to the given directions, and obtain "Mr. T. C. Mits" (Fig. 67). The verification of this figure is left as an exercise.

Let $U = R_e \times R_e$

$$
\begin{aligned}
A &= \{(x,y) \mid 9x^2 + 4y^2 \leq 144\} \\
B &= \{(x,y) \mid y \geq 4x^2 - 1\} \\
C &= \{(x,y) \mid y < 1\} \\
D &= \{(x,y) \mid (x+2)^2 + (y-2)^2 \geq \tfrac{1}{16}\} \\
E &= \{(x,y) \mid (x-2)^2 + (y-2)^2 \geq \tfrac{1}{16}\} \\
F &= \{(x,y) \mid x \leq y^2 - 5 \wedge x < 0\} \\
G &= \{(x,y) \mid x \leq -y^2 + 5 \wedge x > 0\} \\
H &= \{(x,y) \mid (x+2)^2 + 4(y-2)^2 \leq 1\} \\
I &= \{(x,y) \mid (x-2)^2 + 4(y-2)^2 \leq 1\} \\
J &= \{(x,y) \mid 3x + 4y - 12 \geq 0\} \\
K &= \{(x,y) \mid 3x - 4y + 16 \leq 0\} \\
L &= \{(x,y) \mid y \geq x^2 - 4\} \\
M &= \{(x,y) \mid 2y \leq x^2 - 7\} \\
N &= \{(x,y) \mid (x+2)^2 + 4(y-3)^2 = 1 \wedge y > 3\} \\
O &= \{(x,y) \mid (x-2)^2 + 4(y-3)^2 = 1 \wedge y > 3\}
\end{aligned}
$$

Table 1

Set	Color	Feature
a. A	White	Head
b. $B \cap C$	Red	Nose
c. $H \cap D$	White	Left eye
d. $I \cap E$	White	Right eye
e. D'	Black	Left pupil
f. E'	Black	Right pupil
g. $F \cap A'$	White	Left ear
h. $G \cap A'$	White	Right ear
i. $L \cap M$	Red	Mouth
j. $J \cap K \cap A$	Black	Hair
k. N	Black	Left eyebrow
l. O	Black	Right eyebrow

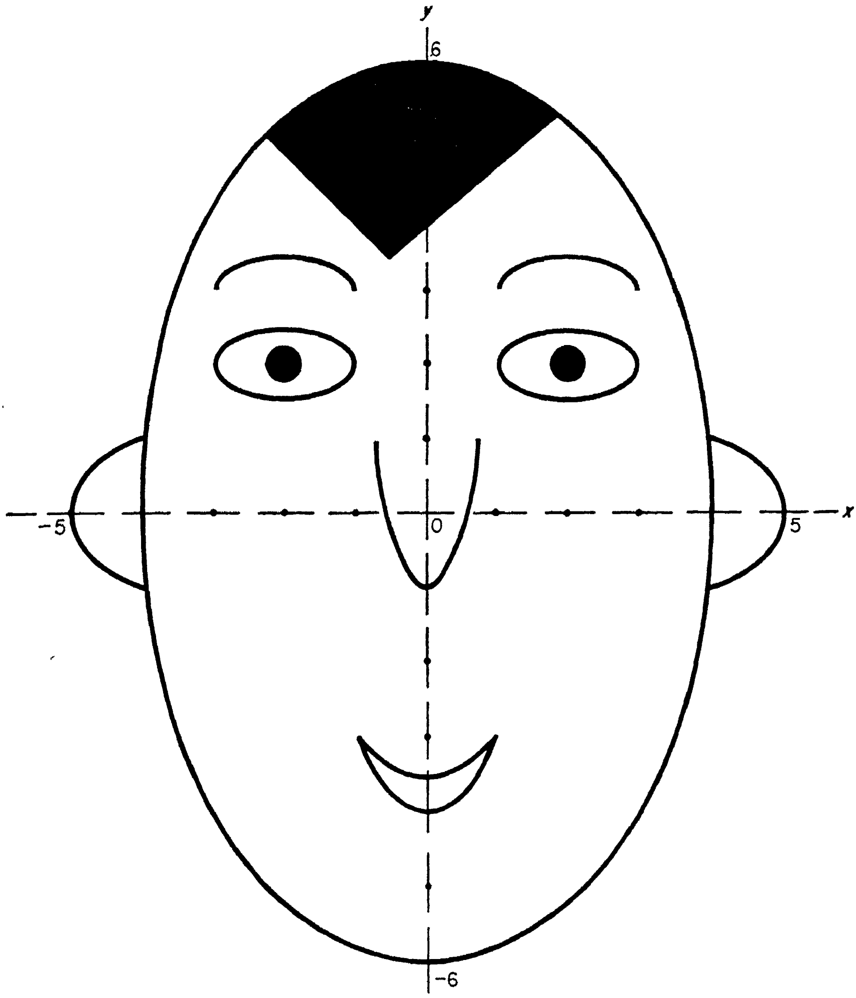


FIG. 67

Exercise 15

The universe used throughout this exercise is $R_e \times R_e$.

1. Graph each of the following sets:

a. $\{(x,y) \mid x + y = 9\}$

b. $\{(x,y) \mid 3x - 5y = 15\}$

c. $\{(x,y) \mid \frac{1}{2}x - \frac{2}{3}y = 2\}$

d. $\{(x,y) \mid 6x - 9y = 12\}$

e. $\{(x,y) \mid x - 2y = 4\}$

f. $\{(x,y) \mid x = 4\}$

g. $\{(x,y) \mid x = 0\}$

h. $\{(x,y) \mid y = 0\}$

2. Graph each of the following:

a. $\{(x,y) \mid x + y > 3\}$

b. $\{(x,y) \mid x - 2y \leq 4\}$

c. $\{(x,y) \mid 3x - 2y > 6\}$

d. $\{(x,y) \mid y > 4\}$

e. $\{(x,y) \mid x \leq 5\}$

f. $\{(x,y) \mid y > 0\}$

g. $\{(x,y) \mid x < 0\}$

h. $\{(x,y) \mid 2x - 3y \geq 4\}$

i. $\{(x,y) \mid y - 3x \leq 3\}$

j. $\{(x,y) \mid 2x + 3y \leq 4\}$

k. $\{(x,y) \mid 2y + 3x \geq 6\}$

3. Graph each of the following:

a. $\{(x,y) \mid x + y = 5 \lor 3x - y = 11\}$

b. $\{(x,y) \mid x + y = 5 \land 3x - y = 11\}$

c. $\{(x,y) \mid x + y = 5 \lor 2x + 2y = 3\}$

d. $\{(x,y) \mid x + y = 5 \land 2x + 2y = 3\}$

e. $\{(x,y) \mid x + y = 3 \lor 2x + 2y = 6\}$

f. $\{(x,y) \mid x + y = 3 \land 2x + 2y = 6\}$

g. $\{(x,y) \mid 2x - y + 7 = 0\} \cap \{(x,y) \mid 3x + 4y - 6 = 0\}$

h. $\{(x,y) \mid 5x - 3y = 15\} \cap \{(x,y) \mid 9y - 15x = -10\}$

i. $\{(x,y) \mid 3x - 2y + 7 = 0\} \cap \{(x,y) \mid 5x + 4y - 14 = 0\}$

j. $\{(x,y) \mid 2x - 3y = 8\} \cap \{(x,y) \mid 3x = \frac{1}{2}(24 + 9y)\}$

k. $\{(x,y) \mid \frac{1}{2}x - \frac{1}{3}y + 2 = 0\} \cap \{(x,y) \mid x - \frac{3}{2} + 2y = 2\frac{1}{2}\}$

4. Graph each of the following:

a. $\{(x,y) \mid x + y > 3\}$

b. $\{(x,y) \mid x - 2y \leq 4\}$

c. $\{(x,y) \mid x \in [-3,5] \text{ and } y \in [0,2]\}$

d. $\{(x,y) \mid x \in [0, \infty[\text{ and } y \in [0, \infty[\}$

e. $\{(x,y) \mid x = -3\} \cap \{(x,y) \mid y \in [0,7]\}$

f. $\{(x,y) \mid \frac{1}{2}x + 3 > 1\} \cap \{(x,y) \mid y < 0\}$

g. $\{(x,y) \mid 2x - y \leq 2 \text{ and } x \geq 0\}$

h. $\{(x,y) \mid x + y \leq 5\} \cap \{(x,y) \mid x - y \leq 3 \land x \geq 0\}$

i. $\{(x,y) \mid x^2 + y^2 \leq 49\}$

j. $\{(x,y) \mid y \geq x^2 + 1\} \cap \{(x,y) \mid y \leq 5\}$

k. $\{(x,y) \mid 9x^2 + 4y^2 \leq 36 \land y \geq 0 \land x \geq 0\}$

l. $\{(x,y) \mid 3x - 2y \leq 6\} \cap \{(x,y) \mid 3x - 2y \geq 9\}$

m. $\{(x,y) \mid 2x + y \geq 2\} \cap \{(x,y) \mid 2x + y \leq 6\}$

n. $\{(x,y) \mid x^2 + y^2 \geq 9 \land x^2 + 9y^2 > 36\}$

o. $\{(x,y) \mid x^2 + y^2 \leq 64 \land x^2 + y^2 \geq 16\}$

p. $\{(x,y) \mid 9x^2 + 16y^2 = 144 \land y = x\}$

q. $\{(x,y) \mid 9x^2 + 16y^2 \leq 144\} \cap \{(x,y) \mid y \leq x\}$

r. $\{(x,y) \mid y^2 = 4x\} \cap \{(x,y) \mid 2x - y = 4\}$

s. $\{(x,y) \mid y^2 \leq 4x \land 2x - y \leq 4\}$

t. $\{(x,y) \mid 9x^2 + 16y^2 \leq 144 \land x^2 - y^2 \geq 4\}$

u. $\{(x,y) \mid 9x^2 + 16y^2 \leq 144\} \cap \{(x,y) \mid x^2 - y^2 \leq 4\}$

5. Let
$$A = \{(x,y) \mid x \geq 0\}$$
$$B = \{(x,y) \mid y \geq 0\}$$
$$C = \{(x,y) \mid x + 2y \geq 6\}$$
$$D = \{(x,y) \mid y - x \geq 0\}$$

Graph each of the following sets:

a. A | *b.* B | *c.* C
d. D | *e.* $A \cap B$ | *f.* $A \cup B$
g. $(A \cap B) \cap C$ | *h.* $[(A \cap B) \cap C] \cap D$

6. If $A = \{(x,y) \mid x^2 + y^2 = 36\}$, graph each of the following subsets:
a. $B = \{(x,y) \mid (x,y) \in A \text{ and } x \geq 0\}$
b. $C = \{(x,y) \mid (x,y) \in A, x \geq 0, \text{ and } y \geq 0\}$
c. $D = \{(x,y) \mid (x,y) \in A \text{ and } y = 0\}$
d. $E = \{(x,y) \in A \mid x < 0 \wedge y < 0\}$

7. Let $A = \{(x,y) \mid x \geq 0\}$
$B = \{(x,y) \mid y \leq 0\}$
$C = \{(x,y) \mid x - y \geq 3\}$
$D = \{(x,y) \mid x^2 + y^2 \leq 9\}$

Graph each of the following sets:

a. C' | *b.* D' | *c.* $B \cap C'$
d. $C' \cup D'$ | *e.* $B \cap C$ | *f.* $C \cap D$
g. $(A \cap B) \cap C$ | *h.* $[(A \cap B) \cap C] \cap D$ | *i.* $(A \cap B) \cap C'$

4.5 RELATIONS INVOLVING ABSOLUTE VALUE

The concept of absolute value affords many opportunities for creating relations which, in turn, describe particular subspaces of the xy plane. To illustrate such situations, the following examples are included.

Example 1. The graph of $D = \{(x,y) \mid |x| + y = 3\}$ is represented in Fig. 68. This follows since the defining condition $|x| + y = 3$ becomes $3 - y = x$ if $x \geq 0$ or $3 - y = -x$ if $x < 0$. Accordingly,

$$D = \{(x,y) \mid x + y = 3 \wedge x \geq 0\} \cup \{(x,y) \mid x - y = -3 \wedge x < 0\}$$

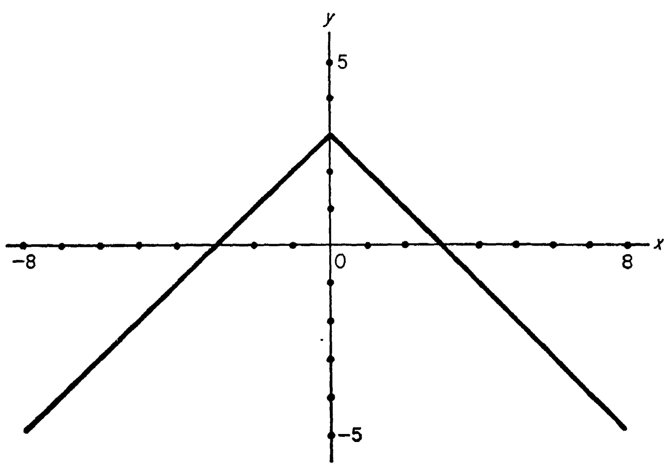


FIG. 68

Example 2. The graph of

$$\begin{aligned} C &= \{(x,y) \mid |x| + y < 3\} \\ &= \{(x,y) \mid x + y < 3 \wedge x \geq 0\} \cup \{(x,y) \mid x - y > -3 \wedge x < 0\} \end{aligned}$$

appears in Fig. 69.

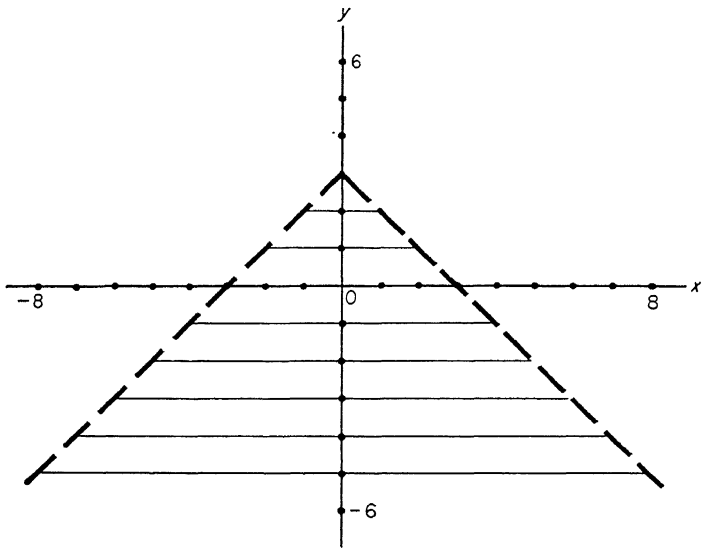


FIG. 69

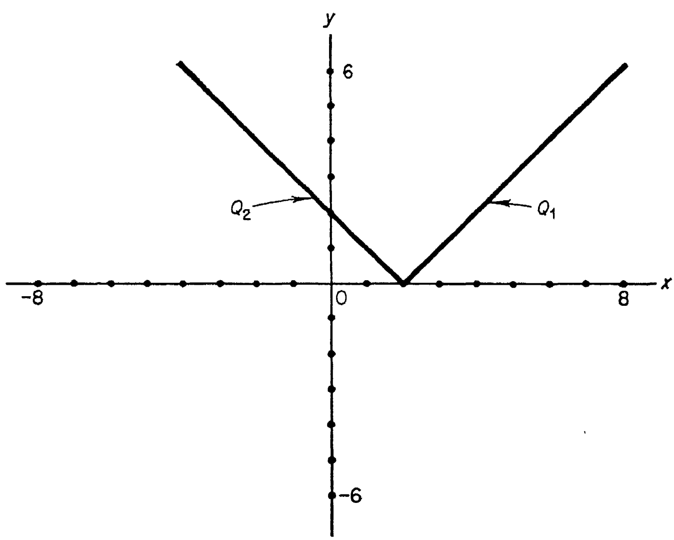


FIG. 70

Example 3. If $Q = \{(x,y) \mid y = |x - 2|\}$, then $Q = Q_1 \cup Q_2$, where $Q_1 = \{(x,y) \mid y = x - 2 \wedge x \geq 2\}$ and $Q_2 = \{(x,y) \mid y = 2 - x \wedge x < 2\}$. The graph of Q appears in Fig. 70.

Example 4. If $M = \{(x,y) \mid |x| + |y| = 4\}$, then (as shown in Fig. 71) $M = M_1 \cup M_2 \cup M_3 \cup M_4$, where

$$M_1 = \{(x,y) \mid x + y = 4 \wedge x \geq 0 \wedge y \geq 0\}$$
$$M_2 = \{(x,y) \mid x - y = 4 \wedge x \geq 0 \wedge y < 0\}$$
$$M_3 = \{(x,y) \mid -x - y = 4 \wedge x < 0 \wedge y < 0\}$$
$$M_4 = \{(x,y) \mid -x + y = 4 \wedge x < 0 \wedge y \geq 0\}$$

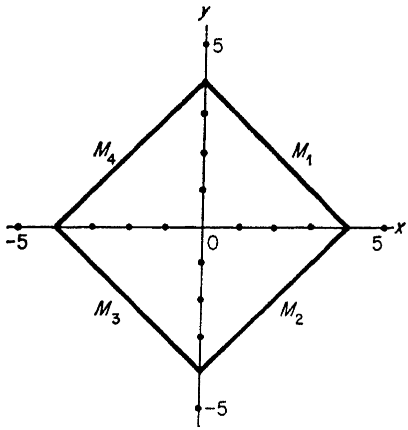


FIG. 71

Exercise 16

The universe throughout this exercise is $R_e \times R_e$.

1. Graph each of the following:

a. $\{(x,y) \mid |y| = x - 2\}$
b. $\{(x,y) \mid |y| = 5\}$
c. $\{(x,y) \mid |x + 2| = 7\}$
d. $\{(x,y) \mid |x - 3| = 2\}$
e. $\{(x,y) \mid |x| - |y| = 4\}$
f. $\{(x,y) \mid |3 - x| = 7\}$
g. $\{(x,y) \mid y = |x^2 - 4|\}$
h. $\{(x,y) \mid y = |x|\}$
i. $\{(x,y) \mid y = 2 - |x|\}$
j. $\{(x,y) \mid |x + y| = 6\}$
k. $\{(x,y) \mid y = x|x - 2|\}$
l. $\{(x,y) \mid |x - y| = 4\}$
m. $\{(x,y) \mid x + |x| = y\}$
n. $\{(x,y) \mid |x - 2| + |y + 1| = 2\}$

2. Graph each of the following:

a. $\{(x,y) \mid |y| - |x| > 1\}$
b. $\{(x,y) \mid |y - x| < 0\}$
c. $\{(x,y) \mid |y - x| > 1\}$
d. $\{(x,y) \mid |y| \geq |x - 2|\}$
e. $\{(x,y) \mid |3 - x| > y\}$
f. $\{(x,y) \mid y > |x| + x\}$
g. $\{(x,y) \mid |x - 2| + |y - 1| \leq 2\}$
h. $\{(x,y) \mid |x| + y \geq 4\}$
i. $\{(x,y) \mid x + |y| < 4\}$
j. $\{(x,y) \mid y > 3 - |x|\}$
k. $\{(x,y) \mid y > |3 - x|\}$

3. Graph each of the following:
a. $\{(x,y) \mid |x| < 2\} \cap \{(x,y) \mid |y| < 3\}$
b. $\{(x,y) \mid y \geq |x| + x\} \cap \{(x,y) \mid y \geq |x| - x\}$
c. $\{(x,y) \mid |2x + y| = 3 \wedge |3x - y| = 5\}$

4. Graph each of the following:
a. $\{(x,y) \mid y = |x^2 - 5x + 6|\}$ b. $\{(x,y) \mid x = |3y^2 - 6y| \wedge y \geq 0\}$

4.6 CONCEPT OF A FUNCTION

A most important unifying idea in mathematics is that of a function. With the advent of a greater appreciation for the use of the language of sets, it now is possible to refine the notions of a function and make its meaning more precise. The concept of a function is basically concerned with a domain D^*, a range R^*, and specific instructions that assign to each element of the domain D^* a corresponding element of the range R^* so as to form ordered pairs. This means that functions are of necessity sets of ordered pairs. By convention, the empty set is not considered a function but is referred to as the void relation. Cartesian products and their subsets are the nuclei from which functions arise.

A relation is a function if it satisfies one important requirement, namely, that of being single-valued; a relation is single-valued if when two ordered pairs are contained with equal first components, then their second components must also be equal. It is noted that the mathematical idea of a function is not destroyed for any relation which possesses two or more ordered pairs with unequal first components, even though their second components are equal. *Every function is a relation, but not every relation is a function.*

Sets of elements, whether these be ordered pairs or otherwise, may be designated by capital letters such as $A, B, \ldots, R, R_1, \ldots$. It will be convenient when discussing functions to represent sets of ordered pairs by lowercase letters such as $f, g, h, \ldots, f_1, f_2, \ldots$, but even in these instances it will not be mandatory. A description of a function in terms of a set of ordered pairs of the form (x,y) is now given. *A function f with domain D^* and range R^* is a nonempty set where for each $x \in D^*$ there is one and only one $y \in R^*$ which combines with x to form each of the (x,y)'s belonging to f.*

A function so defined is referred to as a single-valued function in traditional mathematics. The modern version insists that if a set of ordered pairs satisfies the property of being single-valued, then and only then will it be considered a function. The unspecified element $x \in D^*$ and the unspecified element $y \in R^*$ are called the independent and dependent variables of the function f, respectively. If R^* contains only one element, then f is called a constant function. If $D^* \subseteq R_e$, f is called

a function of a real variable. If $R^* \subseteq R_e$, f is called a real-valued function. In the material which follows, examples illustrate how functions are joined together to form relations and how relations are separated to form functions. These procedures necessitate a clear understanding of the difference between a relation and a function.

Example 1. Let

$$A = \{(2,2),(4,2),(3,1),(5,1)\}$$
$$B = \{(1,1),(2,2),(3,3),(4,4)\}$$
$$C = \{(1,1),(1,3),(2,1),(3,2),(4,1)\}$$

Relations A and B satisfy the requirement of being single-valued and as a consequence are functions. Relation C is not a function, since the ordered pairs (1,1) and (1,3) have equal first components but unequal second components.

4.7 PHRASEOLOGY AND LANGUAGE OF FUNCTIONS

The expression "function of" is not to be confused with the mathematical meaning of "function." As already described, a function is a set of ordered pairs satisfying a special property. However, the phrase y is a "function of" x will imply that ordered pairs (x,y) can be formed possessing the property of single-valuedness. One of the methods used to define a function is to give its domain D^* and rules for obtaining corresponding range values. It is customary to express the rules or defining conditions in the form of formulas or equations. In certain other instances, directions may be given in word sentences which are not expressible in terms of equations and formulas. In all cases, some means must exist whereby the naming of a value in the domain of a function gives rise to the naming of a corresponding value in the range.

For example, the statement "Function $f(x) = \sqrt{25 - 9x^2}$" is an abbreviation for the statement "The function f defined by the equation $f(x) = \sqrt{25 - 9x^2}$." In set notation it is written as

$$f = \{(x,y) \mid x \in R_e \wedge y \in R_e \wedge y = f(x) = \sqrt{25 - 9x^2}\}$$

There are a number of important ideas that should be emphasized here. The notation $f(x)$ is sometimes used in an ambiguous fashion to symbolize a function and also the value of the function. The symbol $f(x)$ is not the function, since f is the function. f represents ordered pairs, while $f(x)$ or y represents the second components of the ordered pairs that make up the function proper. The symbol $f(x)$ by definition is an abbreviation for the value of the "function f at x," or "function f of x." Thus, $f(1) = \sqrt{25 - 9} = 4$ means the value of the function f at 1 is 4.

Similarly, $f(0) = \sqrt{25 - 0} = 5$, and so on. These values $f(1)$ and $f(0)$ are second components of the ordered pairs $(1,f(1)) = (1,4)$ and $(0,f(0)) = (0,5)$. These ordered pairs belong to the function f and not to a function $f(x)$. This ambiguous usage of $f(x)$ has become so embedded in mathematical literature that it seems impossible to eliminate it, and, as a consequence, symbols such as $f(x)$, $g(x)$, and so on have to be read carefully. The statement "The function $f(x) = \sqrt{25 - 9x^2}$" leaves little doubt that the function f is desired, that is, the set of ordered pairs for which $f(x) = \sqrt{25 - 9x^2}$. Accordingly, f is the function; $f(x)$ is the functional or range value of f at x.

Ambiguity appears when a set is described as

$$f(x) = \{(x,y) \mid x \in R_e \wedge y \in R_e \wedge y = f(x) = \sqrt{25 - 9x^2}\}$$

because the "$f(x)$" of the left side does not represent the set of ordered pairs on the right side. The left side should be represented by f and not $f(x)$, and thus ambiguity would be avoided. At best, if $f(x)$ appears as indicated, it must be regarded as a label or a suggestive expression for creating the right side. This is what is understood when expressions such as $\{(x,y) \mid y = f(x) = 9x^2 - 3x + 2\}$ are written as

$$f(x) = \{(x,y) \mid f(x) = 9x^2 - 3x + 2\}$$

or $9x^2 - 3x + 2 = \{(x,y) \mid f(x) = 9x^2 - 3x + 2\}$. Here $9x^2 - 3x + 2$ on the left side suggests the rule from which the corresponding function is created and is distinguishable from $2x$, $\sqrt{9 - x^2}$, etc., which suggest rules that create other functions. The description

$$f = \{(x,f(x)) \mid f(x) = 9x^2 - 3x + 2\}$$

makes a clear distinction between the usage of f and $f(x)$.

In the set $\{(x,y) \mid x \in R_e \wedge y \in R_e \wedge y = f(x) = \sqrt{25 - 9x^2}\}$ the conditions $x \in R_e$ and $y \in R_e$ restrict x and y to the set of real numbers, while the set $\{(x,y) \mid x \in R_e \wedge y = \sqrt{25 - 9x^2}\}$ either could be interpreted the same way or might permit y to be chosen from the set of complex numbers with $x \in R_e$. However, the first interpretation is what is usually meant in both descriptions unless otherwise specified. For this example, the domain D^* and the range R^* are the intervals $[-\frac{5}{3},\frac{5}{3}]$ and $[-5,5]$, respectively.

Example 1. Consider the following phraseology:

a. Graph $y = 1$.
b. Graph $x = 2$.
c. Graph $x^2 + y^2 = 16$.
d. Graph $x^2 + y^2 = -16$.

These may be restated as:

a. $A = \{(x,y) \mid x \in R_e \wedge y = 1\}$

b. $B = \{(x,y) \mid x = 2 \wedge y \in R_e\}$

c. $C = \{(x,y) \mid x \in R_e \wedge y \in R_e \wedge x^2 + y^2 = 16\}$

d. $E = \{(x,y) \mid x \in R_e \wedge y \in R_e \wedge x^2 + y^2 = -16\}$

In each case it is not the equation which is graphed, but the set of ordered pairs which the equation produces.

a. Set A consists of ordered pairs, such as (0,1), (1,1), (2,1), (3,1), and so on. The graph of this set is a line one unit above the x axis and parallel to it, as shown in Fig. 72. Even though all ordered pairs of set

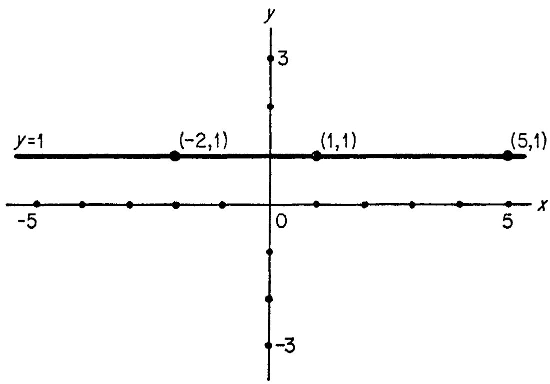


FIG. 72

A have identical second components, their corresponding first components are different. Hence, A is an example of a constant function with domain $]-\infty,\infty[$ and range $\{1\}$.

Because of the nature of the graph, the expressions "line $y = 1$" and "graph of the equation $y = 1$" are used interchangeably. The line constitutes all the points associated with the ordered pairs belonging to A, while $y = 1$ is an equation and nothing more. The function was created because the equation $y = 1$ provided enough information to supply the ordered pairs of A. The line is labeled $y = 1$, but in this sense the equation is a name that identifies the line and distinguishes it from all other graphs of ordered pairs. The set description

$$\{(x,y) \mid x \in R_e \wedge y = 1\}$$

being more awkward to write on the graph, is replaced for convenience by the label $y = 1$.

b. What has been said in part *a* could in the main be repeated for set B. The ordered pairs belonging to B are (2,1), (2,3), (2,0), $(2,\sqrt{2})$, ...

Here B is a relation but not a function. The graph of set B is a line two units to the right of the y axis and parallel to it, as shown in Fig. 73. The domain of B is $\{2\}$, and the range of B is $]-\infty,\infty[$.

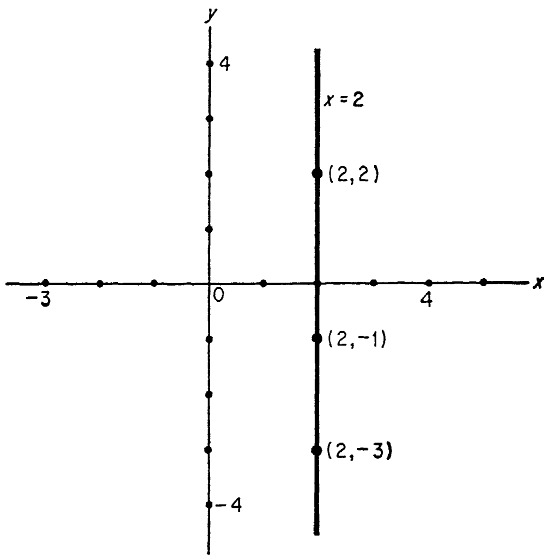


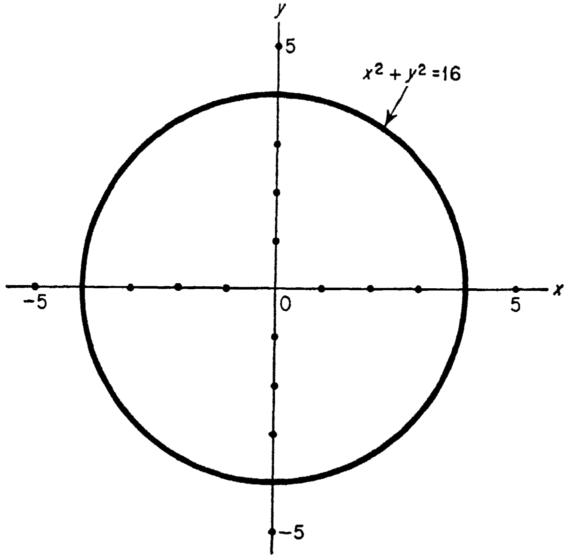
FIG. 73

c. Set C contains ordered pairs such as $(1,\sqrt{15})$, $(1,-\sqrt{15})$, $(0,4)$, $(0,-4)$, ... The ordered pairs contained in C bar it from being a function. The graphical representation of C is a circle with radius of four units and center at (0,0). The expression "circle $x^2 + y^2 = 16$" is used interchangeably with $\{(x,y) \mid x^2 + y^2 = 16\}$. The graph is shown in Fig. 74 and labeled $x^2 + y^2 = 16$. Both the domain and the range of C are represented by the interval $[-4,4]$.

Note that the original equation suggests other equations which include $y = \sqrt{16 - x^2}$ and $y = -\sqrt{16 - x^2}$. The sets suggested by these equations, namely,

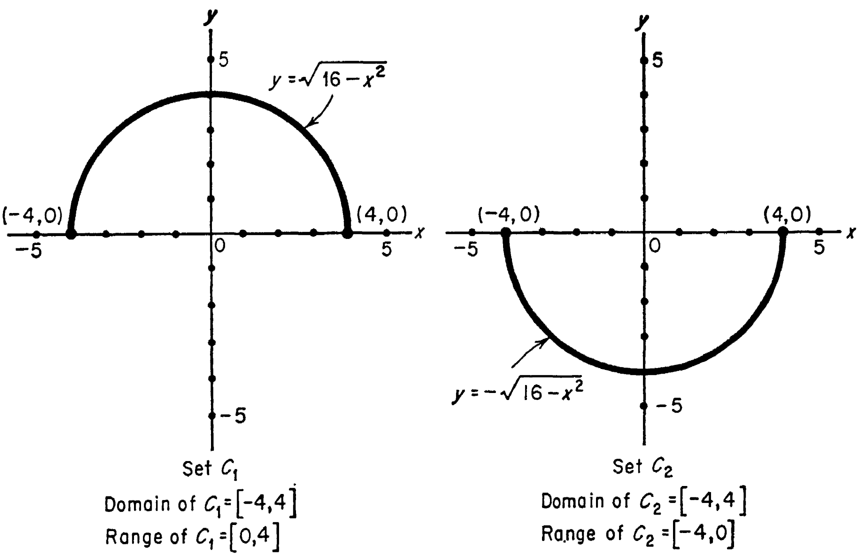
$$C_1 = \{(x,y) \mid y = \sqrt{16 - x^2}\}$$
$$C_2 = \{(x,y) \mid y = -\sqrt{16 - x^2}\}$$

represent functions, and their graphs appear in Fig. 75. It is apparent that the relation C is the union of the functions C_1 and C_2, $C = C_1 \cup C_2$. The domain of C is the union of the "domain of C_1" and the "domain of C_2" or $[-4,4] \cup [-4,4] = [-4,4]$. The range of C is the union of the "range of C_1" and the "range of C_2" or $[0,4] \cup [-4,0] = [-4,4]$.



y
5
$x^2 + y^2 = 16$
−5
0
5
x
−5

FIG. 74



y
5
$y = \sqrt{16 - x^2}$
(−4,0)
(4,0)
−5
0
5
x
−5
Set C_1
Domain of $C_1 = [-4,4]$
Range of $C_1 = [0,4]$
y
5
(−4,0)
(4,0)
−5
0
5
x
$y = -\sqrt{16 - x^2}$
−5
Set C_2
Domain of $C_2 = [-4,4]$
Range of $C_2 = [-4,0]$

FIG. 75

This illustrates how a relation can be separated into the union of two relations, which in this case are both functions. This idea is important for many situations that arise in mathematical environments such as calculus where functions are necessary.

d. The set $E = \{(x,y) \mid x^2 + y^2 = -16\}$ represents the empty set $\emptyset$, since the sum of the squares of two real numbers is never negative. Both the domain and the range of E are represented by the empty set. By convention, set E is the void relation and not a function.

Example 2. Consider the set represented by

$$f = \left\{(x,f(x)) \mid f(x) = \begin{pmatrix} 2 - 3x \wedge x \in [0,1[\\ x \qquad\quad \wedge x \in [1,4] \end{pmatrix}\right\}$$

This set is a function. The open circle at point $(1,-1)$ in Fig. 76 indicates that $(1,-1) \notin f$. This example illustrates that compound conditions are sometimes necessary to define a function.

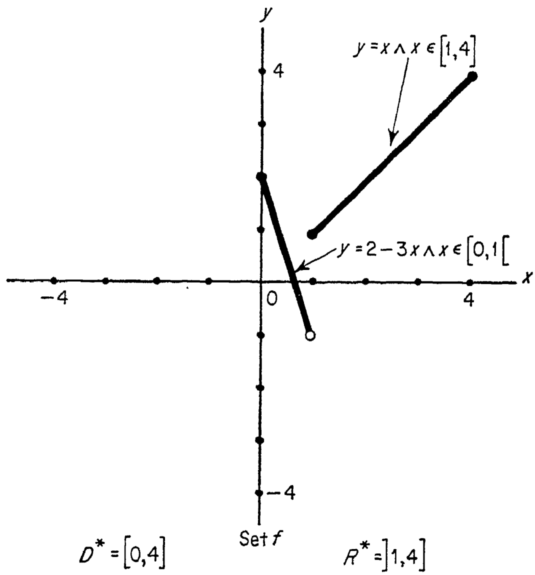


FIG. 76

In many physical environments the term function is used to describe a formula that applies to a particular situation. This is illustrated in the examples that follow.

Example 3. In the word statement "Express the radius of a circle as a function of its area," the phrase "function of" implies that ordered pairs of numbers can be created where the first and second components are

associated with area and radius, respectively. It is implicitly understood that the circle is to have a realistic environment, and thus the area and radius must be represented by positive real numbers. The use of negative real numbers has no significant meaning here, but zero area and zero radius might be included. In this example, the domain D^* is either $]0, \infty[$ or $[0, \infty[$.

If A and r are associated with the area and radius of a circle, respectively, then a well-known formula from geometry states that $A = \pi r^2$ or that $r = \sqrt{A/\pi}$. It is now possible to form the desired function by indicating a domain and applying a rule. Accordingly, for this example, a set of ordered pairs (A,r) can be created that satisfies the requirements of being a mathematical function. In set notation the problem "Express the radius of a circle as a function of its area" can be stated as

$$\{(A,r) \mid A \in]0, \infty[\wedge A = \pi r^2\} \quad \text{or} \quad \left\{(A,r) \mid A \in]0, \infty[\wedge r = \sqrt{\frac{A}{\pi}}\right\}$$

Since $A \in]0, \infty[$ is implicitly understood, the form $\{(A,r) \mid r = \sqrt{A/\pi}\}$ is preferable. The range R^* of this function is $]0, \infty[$.

If the problem had been phrased, "Express the area of a circle as a function of its radius," then the function $\{(r,A) \mid r \in]0, \infty[\wedge A = \pi r^2\}$ would be implied, where r and A are considered to be the domain and range variables, respectively.

Example 4. Let the radius and height of a right circular cylinder be represented by r and h linear units, respectively. If h is constant, the total surface area S is a "function of" r. From geometry,

$$S = f(r) = 2\pi rh + 2\pi r^2$$

square units, which suggests the creation of the function

$$f = \{(r,S) \mid S = f(r) = 2\pi rh + 2\pi r^2\}$$

If the right circular cylinder is given the fixed volume of 100 cubic units, that is, $r^2h = 100$, then f can be restated as

$$\{(r,S) \mid S = f(r) = 200/r + 2\pi r^2\}$$

Both the range and the domain of this function are represented by the interval $]0, \infty[$.

Exercise 17

1. Which of the following sets are functions?

a. $\{(3,1),(4,1)(5,1),(-1,1)\}$ *b.* $\{(2,2),(3,3),(4,4),(5,5)\}$

c. $\{(0,1),(0,2),(1,3),(2,4)\}$ *d.* $\{(1,4)\}$

e. $\{\ \ \} = \emptyset$

2. If $U = R_e \times R_e$, which of the following are functions?
$a.$ $\{(x,y) \mid 2y = 1\}$ $b.$ $\{(x,y) \mid x = -2\}$

3. What functions are suggested by each of the following expressions?
$a.$ $\sqrt{4 - x^2}$ $b.$ -5 $c.$ $\sqrt{9 - 4x^2}$
$d.$ 0 $e.$ $\sqrt{4x^2 - 9}$ $f.$ $2x$
$g.$ $\sqrt{4x^2 + 9}$ $h.$ $-x - 1$

4. What functions do we obtain in each of the following?
$a.$ Domain: $D^* = \{4,1,2,0\}$.
Rule: (1) Choose an element in the domain.
(2) Square and subtract 3.
$b.$ Domain: $D^* = R_e =]-\infty, \infty[$.
Rule: (1) Multiply the element chosen by $(-\frac{1}{2})$.
(2) Add 3 to the result in step (1).
(3) Cube the result in step (2).
Tabulate at least five ordered pairs.
$c.$ Domain: $D^* = R_e =]-\infty, \infty[$.
Rule: (1) For all numbers of R_e in the interval $]-1,4[$, double the number chosen.
(2) For all numbers in R_e in the interval $[7,10]$, divide the number chosen by 2 and to the result add 6.
(3) For all other numbers chosen not in the intervals described in steps (1) and (2), subtract 4.
Tabulate at least four ordered pairs in each of the intervals described.
$d.$ Domain: $D^* =]0, \infty[$, which represent numbers describing the edge of a cube.
Rule: (1) Choose a number in D^*.
(2) Cube it.
Form the function and describe the geometrical significance of the range values $\in R^*$.
$e.$ Domain: The telephone numbers of all the people listed on page 200 of your local telephone directory.
Rule: For each listed telephone number, associate the corresponding name.
Will this always be a function, or is it possible for the same telephone number to be assigned to two distinct names on the same page? (Do not tabulate the ordered pairs.)

5. If x represents the domain variable and is chosen from the number interval $]0, \infty[$, state in words the rule of the function described by each of the following:
$a.$ $\dfrac{3x - 1}{2}$ $b.$ $x^2 + 4$ $c.$ $\sqrt{4 - x^2}$
$d.$ $\dfrac{x + 4}{x}$ $e.$ $-\sqrt[3]{9 + x^2}$

6. What is meant by each of the following?
$a.$ The function $y = 3x^2$
$b.$ The function $f(x) = 2x^2 + 3x$
$c.$ The function $2x - 1$
$d.$ The constant function -2

7. If $U = R_e \times R_e$, state the domain and the range of each of the following functions:
$a.$ $f = \{(x,y) \mid y = \sqrt{4x - 1}\}$ $b.$ $g = \{(x,y) \mid y = \sqrt{1 - 4x}\}$
$c.$ $h = \{(x,h(x)) \mid h(x) = \sqrt{1 - 4x^2}\}$ $d.$ $k = \{(y,k(y)) \mid k(y) = \sqrt{2 + y}\}$

8. Describe both verbally and graphically what is meant by each of the following:
a. The line $3x - 2y = 7$
b. The ellipse $4x^2 + 9y^2 = 36$
c. The parabola $x^2 = 4y$
d. The hyperbola $xy = 10$
e. The point (2,8)

9. How would you label the graphs of each of the following sets?
a. $f = \{(x,y) \mid 2x - y = 4\}$ *b*. $g = \{(x,g(x)) \mid g(x) = x^2 + 4\}$
c. $h = \left\{(x,y) \mid y = \begin{pmatrix} 2x \wedge x \in [-2,4] \\ \dfrac{3x+1}{2} \wedge x \in]-4,-2] \end{pmatrix}\right\}$

10. Discuss and represent each of the following statements as a symbolic expression in set notation:
a. The volume of a sphere as a function of the radius.
b. The radius of a sphere as a function of the volume.
c. The volume of a parallelepiped as a function of the width w, if the length is twice the depth and the depth is 4 inches less than the width.
d. The volume of a gas as a function of the pressure p.
e. The area of an equilateral triangle as a function of the length of a side s.
f. The simple interest on \$100 at 5 per cent as a function of the time t.
g. The surface area of a sphere as a function of its radius.
h. The radius of a sphere as a function of its surface area.
i. If $A = (3 + x)/(3 - x)$ and $B = (3 - x)/(3 + x)$ determine the expression $A^2 + 2AB + B^2$ as a function of x.
j. A sphere of radius r is concentric with a sphere $2r$. Express the volume of the spherical shell as a function of r.

As an application to physical situations, set up the appropriate functions for Problems 11 to 18.

11. A strip of nickel 200 inches long and 16 inches wide is to be made into a rain gutter by turning up the edges to form a trough with a rectangular cross section. If the bent-up edge is x inches, express the volume of the trough as a function of x.

12. A gardener wishes to fence a rectangular garden along a straight river requiring no fence. He has enough wire to build a fence 200 yards long. If the side bordering on the river is represented by the variable x, express the area of the garden as a function of x.

13. At noon a car A traveling west at the rate of 25 mph is 20 miles directly south of a car B traveling north at the rate of 30 mph. Express the distance between the two cars as a function of the time t, where t is the number of hours after 12 o'clock. Using the result obtained, find the distance between the cars at 3 P.M. and at 5:15 P.M.

14. If a Norman window (a rectangle surmounted by a semicircle) has a perimeter of 100 inches, and if the side which is not the diameter of the semicircle is represented by x, express the area of the window as a function of x.

15. If the sum of two numbers is 200, and one of the numbers is represented by x, express (*a*) their product as a function of x, (*b*) the sum of their squares as a function of x, and (*c*) the sum of their reciprocals as a function of x.

16. If an open-top tomato can holds 108 cubic inches, and if x represents the number of inches in the radius of the top, express the area of the surface of the can as a function of x.

17. A wire 100 feet long is cut in two parts; one part is bent into the sides of a square, and the other into the circumference of a circle. If the perimeter of the square

is represented by p, express the sum of the areas of both the square and the circle as a function of p.

18. From each corner of a square sheet of cardboard 20 inches on a side, a square of side x inches is cut. The edges of the sheet are then turned up to make a box. Express the volume V of the box as a function of x.

19. If $f(n)$ is the number of prime numbers less than or equal to any positive integer n, determine $f(5)$, $f(50)$, $f(73)$, and $f(79)$.

4.8 THE OPERATIONS OF UNION AND INTERSECTION AS APPLIED TO RELATIONS AND FUNCTIONS

The circle, the ellipse, the hyperbola, and the parabola afford effective illustrations for the use of set operations as applied to relations and functions.

Example 1. A standard form for the equation of a circle with center at the point (h,k) and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2$$

Let $S = \{(x,y) \mid x^2 + y^2 - 4x - 10y - 20 = 0\}$.

a. The equation $x^2 + y^2 - 4x - 10y - 20 = 0$ can be restated as $(x - 2)^2 + (y - 5)^2 = 49$. Hence, the center of the circle is at $(2,5)$ and its radius is 7.

b. The graph of S is labeled in Fig. 77 as $x^2 + y^2 - 4x - 10y - 20 = 0$.

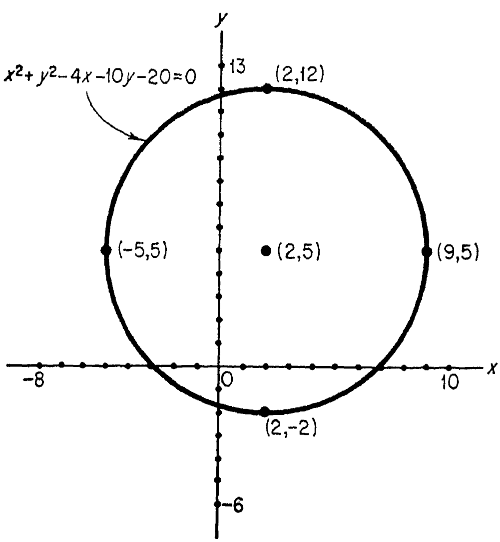


FIG. 77

c. Solving for y in terms of x, we find

$$y = 5 + \sqrt{49 - (x-2)^2} \quad \text{and} \quad y = 5 - \sqrt{49 - (x-2)^2}$$
$$= 5 + \sqrt{45 - x^2 + 4x} \qquad\qquad = 5 - \sqrt{45 - x^2 + 4x}$$

Then

$$S_1 = \{(x,y) \mid y = 5 + \sqrt{45 - x^2 + 4x}\}$$
$$S_2 = \{(x,y) \mid y = 5 - \sqrt{45 - x^2 + 4x}\}$$

and $S = S_1 \cup S_2$.

d. The graphs of S_1 and S_2 appear in Fig. 78 and are labeled accordingly. The graphs of S_1 and S_2 represent sets of ordered pairs which

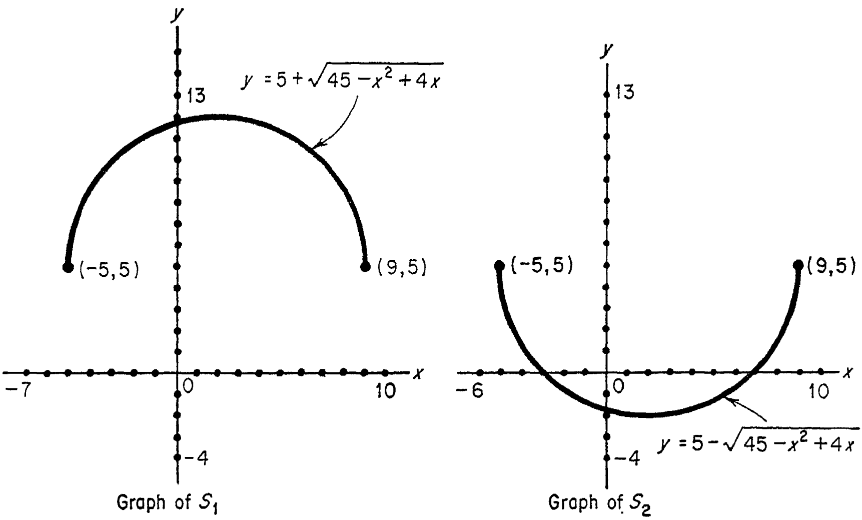


FIG. 78

possess the property of being single-valued. Hence, these relations are functions and their union is the original set S.

e. The domains of S, S_1, and S_2 are each equal to the interval $[-5,9]$. The range of S is the interval $[-2,12]$; the ranges of S_1 and S_2 are the intervals $[5,12]$ and $[-2,5]$, respectively. The range of S is the union of the ranges of S_1 and S_2, namely $[5,12] \cup [-2,5] = [-2,12]$. Note that $S_1 \cap S_2 = \{(-5,5),(9,5)\}$.

Example 2. A standard form for the defining condition of an x ellipse with the center at (h,k) and a and b the semimajor and semiminor axes,

respectively, is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Let $S = \{(x,y) \mid 4x^2 + 9y^2 - 8x + 90y + 193 = 0\}$.

a. The equation $4x^2 + 9y^2 - 8x + 90y + 193 = 0$ can be restated as $(x-1)^2/9 + (y+5)^2/4 = 1$. The ellipse has its center at $(1,-5)$ with a and b equal to 3 and 2, respectively.

b. The graph of S is illustrated in Fig. 79.

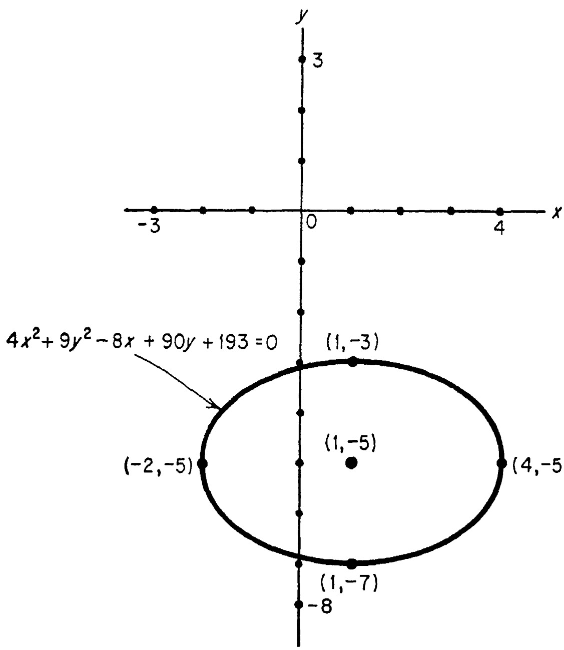


FIG. 79

c. Solving for y in terms of x, we find

$$\begin{aligned} y &= -5 + \tfrac{2}{3}\sqrt{9-(x-1)^2} \\ &= -5 + \tfrac{2}{3}\sqrt{8 - x^2 + 2x} \end{aligned} \qquad \text{and} \qquad \begin{aligned} y &= -5 - \tfrac{2}{3}\sqrt{9-(x-1)^2} \\ &= -5 - \tfrac{2}{3}\sqrt{8 - x^2 + 2x} \end{aligned}$$

This results in the sets

$$S_1 = \{(x,y) \mid y = -5 + \tfrac{2}{3}\sqrt{8 - x^2 + 2x}\}$$
$$S_2 = \{(x,y) \mid y = -5 - \tfrac{2}{3}\sqrt{8 - x^2 + 2x}\}$$

where $S = S_1 \cup S_2$.

d. The graphs of S_1 and S_2 appear in Fig. 80. The graphs of S_1 and S_2 represent sets of ordered pairs which are functions.

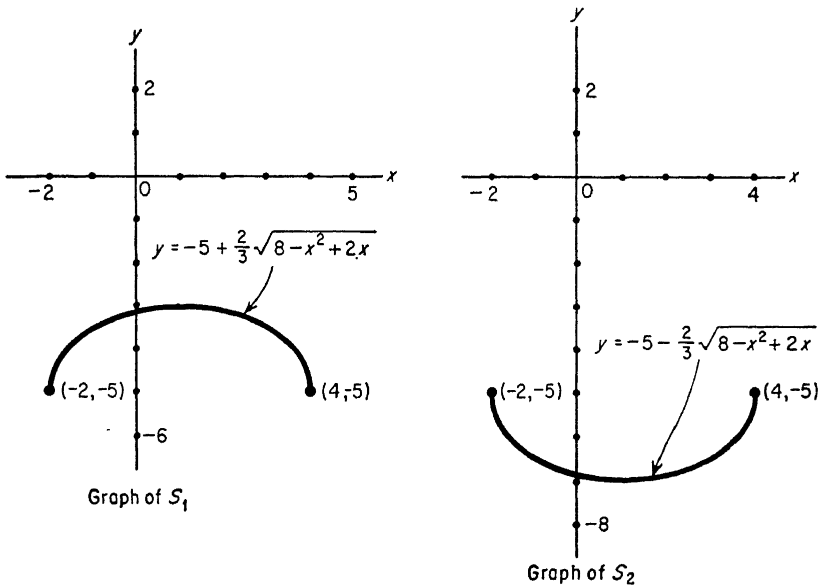


FIG. 80

e. The domains of S, S_1, and S_2 are each equal to the interval $[-2,4]$. The range of S is the union of the ranges of S_1 and S_2, namely,

$$[-5,-3] \cup [-7,-5] = [-7,-3]$$

Note that $S_1 \cap S_2 = \{(-2,-5),(4,-5)\}$.

Example 3. A standard form for the equation of an x hyperbola with center at (h,k) and a and b the semitransverse and semiconjugate axes, respectively, is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Let $S = \{(x,y) \mid 4x^2 - 16y^2 + 24x + 64y - 92 = 0\}$.

a. The equation $4x^2 - 16y^2 + 24x + 64y - 92 = 0$ can be restated as $(x+3)^2/16 - (y-2)^2/4 = 1$. This hyperbola has its center at $(-3,2)$ and a semitransverse axis of 4 and a semiconjugate axis of 2.

b. The graph of S is illustrated in Fig. 81.

c. Solving for y in terms of x, we find

$$y = 2 + \tfrac{1}{2}\sqrt{(x+3)^2 - 16} \qquad \text{and} \qquad y = 2 - \tfrac{1}{2}\sqrt{(x+3)^2 - 16}$$
$$= 2 + \tfrac{1}{2}\sqrt{x^2 + 6x - 7} \qquad\qquad\qquad = 2 - \tfrac{1}{2}\sqrt{x^2 + 6x - 7}$$

This results in the sets

$$S_1 = \{(x,y) \mid y = 2 + \tfrac{1}{2}\sqrt{x^2 + 6x - 7}\}$$
$$S_2 = \{(x,y) \mid y = 2 - \tfrac{1}{2}\sqrt{x^2 + 6x - 7}\}$$

where $S = S_1 \cup S_2$.

d. The graphs of S_1 and S_2 appear in Fig. 82. The graphs of S_1 and S_2 represent sets of ordered pairs which are functions.

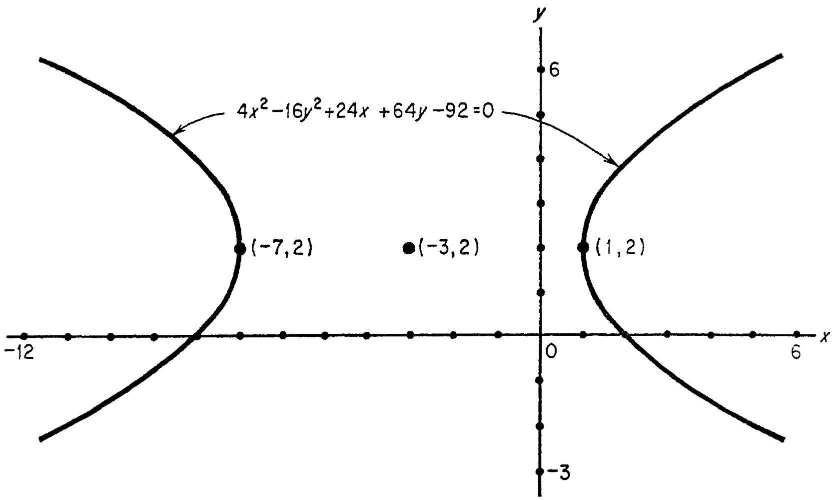


FIG. 81

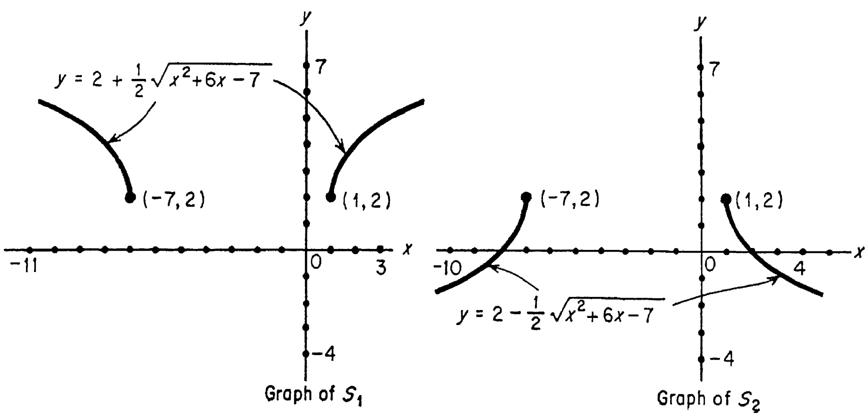


FIG. 82

c. The domains of S, S_1, and S_2 are each equal to the union of the intervals $]-\infty,-7] \cup [1,\infty[$ or $\{x \in R_e \mid x \notin]-7,1[\}$. The range of S is the interval $]-\infty,\infty[$. The range of S is the union of the ranges of S_1 and S_2, namely, $[2,\infty[\cup]-\infty,2] =]-\infty,\infty[$. Note that

$$S_1 \cap S_2 = \{(-7,2),(1,2)\}$$

Example 4. A standard form for the equation of an x parabola where (h,k) represents its vertex and $|4a|$ its focal width is given by

$$(y - k)^2 = 4a(x - h)$$

Let $S = \{(x,y) \mid y^2 - x - 4y + 4 = 0\}$.

a. The equation $y^2 - x - 4y = 0$ can be rewritten as $(y - 2)^2 = x$. The parabola has its vertex at $(0,2)$ and has a focal width of 1.

b. The graph of S is indicated in Fig. 83.

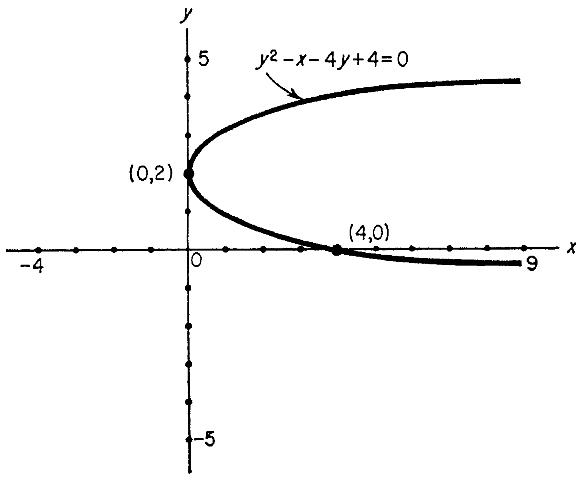


FIG. 83

c. Solving for y in terms of x, we find

$$y = 2 + \sqrt{x} \qquad \text{and} \qquad y = 2 - \sqrt{x}$$

This results in the sets

$$S_1 = \{(x,y) \mid y = 2 + \sqrt{x}\}$$
$$S_2 = \{(x,y) \mid y = 2 - \sqrt{x}\}$$

where $S = S_1 \cup S_2$.

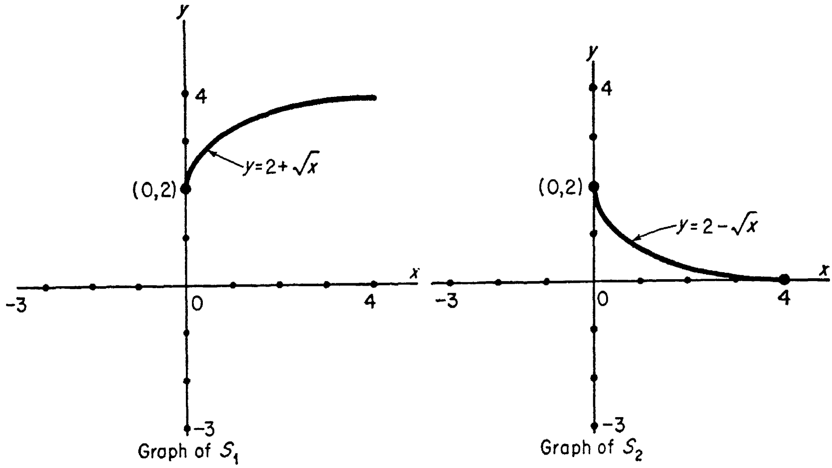


FIG. 84

d. The graphs of S_1 and S_2 are shown in Fig. 84. The graphs of S_1 and S_2 represent sets of ordered pairs which are functions.

e. The domains of S, S_1, and S_2 are each equal to the interval $[0,\infty[$. The range of S is the union of the ranges of S_1 and S_2, namely,

$$[2,\infty[\ \cup\ ]-\infty,2] = \ ]-\infty,\infty[$$

Note that $S_1 \cap S_2 = \{(0,2)\}$.

Example 5. Many functions are described by defining conditions such as $y = f(x) = ax + b$ or $y = f(x) = ax^2 + bx + c$ where $a,b,c \in R_e$ and $a \neq 0$. Their graphical representation in a $R_e \times R_e$ space is either a line or a parabola. Such functions are referred to as linear and quadratic functions, respectively. For example, the equations $y = f(x) = 2x - 3$ and $y = f(x) = 3x^2 - 2x + 1$ define linear and quadratic functions, respectively.

Exercise 18

Using the following methods, discuss Problems 1 to 12 as Examples 1 through 4 of Section 4.8 were discussed.

a. Reduce the equation to a standard form.

b. Graph a relation S.

c. Solve the equation for y in terms of x and show that S can be associated with two sets S_1 and S_2.

d. Graph S_1 and S_2 and show that they are functions.

e. State the domain and range of S and show how these are obtained as unions of the domains and ranges of S_1 and S_2.

Circles:

1. $S = \{(x,y) \mid x^2 + y^2 + 6x - 4y + 4 = 0\}$
2. $S = \{(x,y) \mid x^2 + y^2 + 8x + 10y + 5 = 0\}$
3. $S = \{(x,y) \mid x^2 + y^2 + 12y - 13 = 0\}$

x *Ellipses:*

4. $S = \{(x,y) \mid 4x^2 + 16y^2 + 24x - 64y - 64 = 0\}$
5. $S = \{(x,y) \mid x^2 + 8y^2 - 6x - 16y + 9 = 0\}$
6. $S = \{(x,y) \mid 3x^2 + 5y^2 + 24x - 50y + 53 = 0\}$

x *Hyperbolas:*

7. $S = \{(x,y) \mid 9x^2 - 4y^2 + 18x + 24y - 63 = 0\}$
8. $S = \{(x,y) \mid x^2 - y^2 - 8x + 8y - 4 = 0\}$
9. $S = \{(x,y) \mid x^2 - y^2 + 4x + 16y - 64 = 0\}$

x *Parabolas:*

10. $S = \{(x,y) \mid y^2 - x - 3y + 4 = 0\}$
11. $S = \{(x,y) \mid y^2 - x - 6y + 9 = 0\}$
12. $S = \{(x,y) \mid y^2 - x - 4 = 0\}$

The graph of each of the relations in Problems 13 to 17 is either a point, the null set, or two lines. For example, let $S = \{(x,y) \mid x^2 + y^2 + 4x - 8y + 20 = 0\}$. Here $x^2 + y^2 + 4x - 8y + 20 = 0$ can be restated as $(x + 2)^2 + (y - 4)^2 = 0$, which is satisfied only by the ordered pair $(-2,4)$. Hence, $S = \{(-2,4)\}$, which graphically is the point at $(-2,4)$. Note that the ordered pair $(-2,4)$ is not equal to the set S—$(-2,4) \neq \{(-2,4)\}$—but $(-2,4) \in \{(-2,4)\}$. Is S a function? Why?

Discuss and graph each of the relations in Problems 13 to 17.

13. $S = \{(x,y) \mid x^2 + 4y^2 + 6x + 8y + 13 = 0\}$
14. $S = \{(x,y) \mid x^2 + 4y^2 + 6x + 8y + 25 = 0\}$
15. $S = \{(x,y) \mid x^2 - y^2 - 4x + 6y - 5 = 0\}$
16. $S = \{(x,y) \mid 9x^2 - y^2 - 12x - 4y = 0\}$
17. $S = \{(x,y) \mid x^2 + y^2 + 9x - 3y + 40 = 0\}$

18. Graph each of the following relations and determine whether it is a function:

a. $S_1 = \{(x,y) \mid 2x - 4y = 1\}$
b. $S_2 = \{(x,y) \mid x = 1\}$
c. $S_3 = \{(x,y) \mid 2y = 1\}$
d. $S_4 = \{(x,y) \mid y = \sqrt{4 - x^2}\}$
e. $S_5 = \{(x,y) \mid y = -\sqrt{4 - x^2}\}$
f. $S_6 = \{(x,y) \mid y = \sqrt{2x}\}$
g. $S_7 = \{(x,y) \mid y = -\sqrt{2x}\}$
h. $S_8 = \{(x,y) \mid y = \sqrt{x^2 - 4}\}$
i. $S_9 = \{(x,y) \mid y = -\sqrt{x^2 - 4}\}$
j. $S_{10} = \{(x,y) \mid y^2 = 2x\}$
k. $S_{11} = \{(x,y) \mid x^2 + y^2 = 4\}$
l. $S_{12} = \{(x,y) \mid x^2 - y^2 = 4\}$

19. Using the indicated relations of Problem 18, discuss the following:

a. $S_1 \cup S_2$
b. $S_2 \cup S_3$
c. $S_1 \cap S_2$
d. $S_2 \cap S_3$
e. $S_4 \cup S_5$, and compare this with the graph of S_{11}
f. $S_6 \cup S_7$, and compare this with the graph of S_{10}
g. $S_8 \cup S_9$, and compare this with the graph of S_{12}

20. Specify the domain and range for each of the relations in Problem 18.

21. Let $A = \{(x,y) \mid 6x - 4y - 8 = 0\}$
$B = \{(x,y) \mid 2x - 3y + 4 = 0\}$
$C = \{(x,y) \mid 2y - 3x + 1 = 0\}$
$E = \{(x,y) \mid y - 7 = 0\}$
$F = \{(x,y) \mid x + 4 = 0\}$

Solve each of the following by a graphical procedure and/or an algebraic procedure. Compare the results obtained.

a. $A \cup B$ *b.* $A \cap B$ *c.* $B \cup C$
d. $B \cap C$ *e.* $C \cup E$ *f.* $C \cap E$
g. $A \cup E$ *h.* $E \cap F$

If $U = R_e \times R_e$, solve Problems 22 through 25 algebraically and/or graphically.

22. $\{(x,y) \mid y^2 = 2x\} \cap \{(x,y) \mid 3x - y = 4\}$
23. $\{(x,y) \mid xy = 6\} \cap \{(x,y) \mid 3x - 2y = 6\}$
24. $\{(x,y) \mid x^2 + 2y^2 = 9\} \cap \{(x,y) \mid 3x^2 + 5y^2 = 25\}$
25. $\{(x,y) \mid 5x^2 + 2y^2 = 13\} \cap \{(x,y) \mid 4x^2 - 7y^2 = 19\}$

If $(x,y,z) \in R_e \times R_e \times R_e$, solve Problems 26 through 28 algebraically.

26. $\{(x,y,z) \mid 4x - 2y + 2z = 10 \wedge 5x + y - z = 1 \wedge 3x + 3y - 2z = -7\}$
27. $\{(x,y,z) \mid 2x + y - 3z = 13 \wedge 3x + 2y - 3z = 36 \wedge 3y + 4z = 1\}$
28. $\left\{(x,y,z) \,\middle|\, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 12 \wedge \frac{3}{x} - \frac{4}{y} + \frac{5}{z} = 18 \wedge \frac{5}{x} - \frac{3}{y} + \frac{2}{z} = 13\right\}$

If $U = C \times C$, where C is the set of complex numbers, solve Problems 29 and 30.

29. $\{(x,y) \mid 5x^2 + 2y^2 = 13\} \cap \{(x,y) \mid 4x^2 - 7y^2 = 19\}$
30. $\{(x,y) \mid 5xy - 9x^2 = 1 \wedge y^2 - 2xy = 0\}$

31. Solve both algebraically and graphically and compare the results:
a. $\{(x,y) \mid y = x^2 - 2x - 8 \wedge 4x - y - 17 = 0\}$
b. $\{(x,y) \mid y = x^2 - 2x - 8 \wedge 2x - y - 3 = 0\}$
c. $\{(x,y) \mid y = x^2 - 2x - 8 \wedge x - 2 = 0\}$
(NOTE: Parts *a* and *b* involve linear and quadratic functions.)

4.9 SUM, DIFFERENCE, PRODUCT, AND QUOTIENT FUNCTIONS

The combining of numbers through the operations of addition, subtraction, multiplication, and division results in other numbers called their sum, difference, product, and quotient, respectively. Correspondingly, functions may be combined to obtain sum, difference, product, and quotient functions. The functions discussed in this section will be restricted to $R_e \times R_e$.

If f and g are functions with ordered pairs of the form $(x,f(x))$ and $(x,g(x))$, respectively, then Table 1 summarizes the four different ways of combining f and g to obtain a third function. If $U = R_e \times R_e$, the domain of the sum, difference, product, and quotient functions is made

Table 1

Notation	Name of function	Domain variable	Range variable	Domain of function
$f+g$	Sum	x	$(f+g)(x) = f(x) + g(x)$	$D_f^* \cap D_g^*$
$f-g$	Difference	x	$(f-g)(x) = f(x) - g(x)$	$D_f^* \cap D_g^*$
fg	Product	x	$fg(x) = [f(x)][g(x)]$	$D_f^* \cap D_g^*$
$\frac{f}{g}$	Quotient	x	$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$D_f^* \cap D_g^*$ and $g(x) \neq 0$

up of those real numbers for which the combining of $f(x)$ and $g(x)$ has meaning.

Example 1. Let f be the function defined by $f(x) = 2x$ and let g be the function defined by $g(x) = x^2 - 3$. Then

$$f = \{(x,f(x)) \mid f(x) = 2x\} = \{(x,2x) \mid f(x) = 2x\}$$

and

$$g = \{(x,g(x)) \mid g(x) = x^2 - 3\} = \{(x, x^2 - 3) \mid g(x) = x^2 - 3\}$$

Hence
$$\begin{aligned} f+g &= \{(x,f(x)+g(x)) \mid f(x)+g(x) = x^2 + 2x - 3\} \\ f-g &= \{(x,f(x)-g(x)) \mid f(x)-g(x) = 3 + 2x - x^2\} \\ fg &= \{(x,[f(x)][g(x)]) \mid [f(x)][g(x)] = 2x^3 - 6x\} \\ \frac{f}{g} &= \left\{\left(x,\frac{f(x)}{g(x)}\right) \middle| \frac{f(x)}{g(x)} = \frac{2x}{x^2-3} \wedge |x| \neq \sqrt{3}\right\} \end{aligned}$$

The respective domains and ranges of these functions are:

$$\begin{aligned} D_{f+g}^* &= R_e & R_{f+g}^* &= [-4,\infty[\\ D_{f-g}^* &= R_e & R_{f-g}^* &=]-\infty,4] \\ D_{fg}^* &= R_e & R_{fg}^* &= R_e \\ D_{f/g}^* &= \{x \in R_e \mid |x| \neq \sqrt{3}\} & R_{f/g}^* &= R_e \end{aligned}$$

Example 2. To sketch $f+g$ of Example 1, f and g should first be graphed on the same set of axes. Then for selected values of the domain the corresponding $f(x)$ and $g(x)$ values could be added to obtain the $(f+g)(x)$ value. For example, in Fig. 85 $(f+g)(-1)$ is obtained by adding line segment AB to itself, since line segment AB represents both $g(-1)$ and $f(-1)$. Therefore, line segment AD represents

$$f(-1) + g(-1) = (f+g)(-1)$$

which is the second component of the ordered pair of $f+g$ associated with point D. Similarly, other points of the graph of $f+g$ can be

determined so as to eventually sketch this function. The dotted line of Fig. 85 represents the graph of $f + g$.

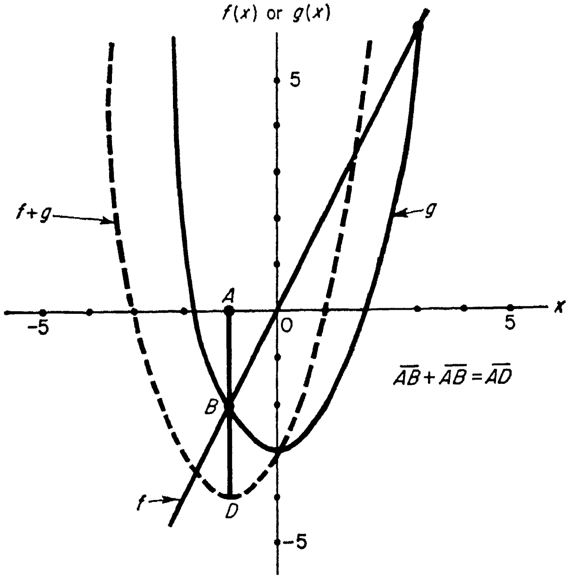


FIG. 85

Another method of obtaining elements of the function $f + g$ is to perform the addition of $f(x)$ and $g(x)$ as shown in Table 2. The table provides enough points to sketch $f + g$.

Table 2

x	$f(x)$	$g(x)$	$f(x) + g(x)$	Ordered pairs
-3	-6	6	$-6 + 6 = 0$	$(-3,0)$
-2	-4	1	$-4 + 1 = -3$	$(-2,-3)$
-1	-2	-2	$-2 - 2 = -4$	$(-1,-4)$
0	0	-3	$0 - 3 = -3$	$(0,-3)$
1	2	-2	$2 - 2 = 0$	$(1,0)$
2	4	1	$4 + 1 = 5$	$(2,5)$
3	6	6	$6 + 6 = 12$	$(3,12)$

4.10 INVERSE FUNCTIONS

A function f is a set of ordered pairs, no two of which have the same first component with different second components. If the function f has the additional characteristic that no two of its ordered pairs have the

same second component with different first components, then the function f has an inverse function, designated as f^{-1}.

A defining condition for the inverse relation R^{-1} is determined from the original defining condition of the given relation by interchanging the variables x and y. Hence, if

$$R = \{(x,y) \mid x^2 = y - 1\}$$

then

$$R^{-1} = \{(x,y) \mid y^2 = x - 1\}$$

Further, the domain of R is identical to the range of R^{-1}, and the range of R is identical to the domain of R^{-1}. If a given relation is a function, it cannot be assumed that the inverse relation will also be a function. The inverse must be examined to determine whether the requirements of a function are satisfied.

Example 1*a*. If $f = \{(2,1)(3,2),(4,3)\}$, then $f^{-1} = \{(1,2),(2,3),(3,4)\}$. Here both f and f^{-1} are functions.

b. If $f = \{(5,-2),(6,-2),(3,4)\}$, then $f^{-1} = \{(-2,5),(-2,6),(4,3)\}$. Here f is a function, but f^{-1} is not a function.

c. If $f = \{(1,5),(1,3),(1,-2)\}$, then $f^{-1} = \{(5,1),(3,1),(-2,1)\}$. Here f is not a function, but f^{-1} is a function.

d. If $f = \{(1,3),(1,-2),(3,-2)\}$, then $f^{-1} = \{(3,1),(-2,1),(-2,3)\}$. Here f and f^{-1} are both relations but not functions.

Example 2. If

$$f = \{(x,y \mid y - 2 = |x| - x\}$$

then

$$f^{-1} = \{(x,y) \mid x - 2 = |y| - y\}$$

The graphs of f and f^{-1} in Fig. 86 show that f is a function, while f^{-1}

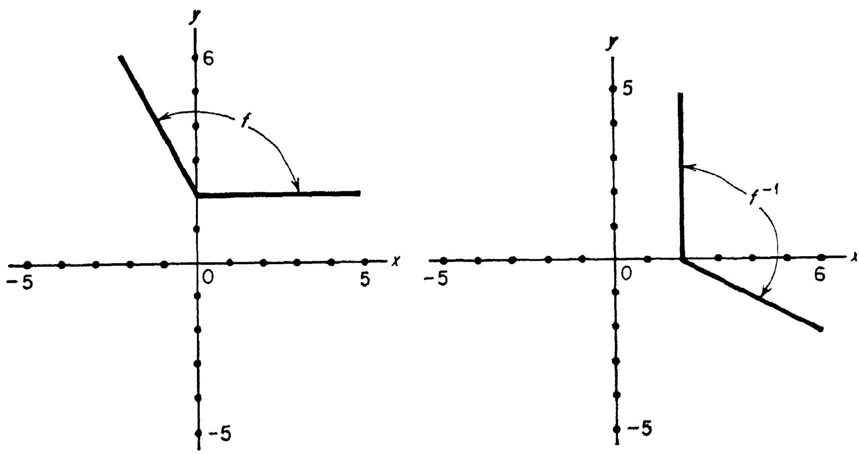


FIG. 86

is a relation but not a function. Here f^{-1} contains ordered pairs such as (2,5),(2,4) and others which have equal first components but different second components. The interval $]-\infty, \infty[$ represents both the domain of f and the range of f^{-1}, while the interval $[2, \infty[$ represents both the range of f and the domain of f^{-1}.

Example 3. The graph of a function f is the set of all those points in the xy plane corresponding to the ordered pairs belonging to f. Consequently, a function describes a subspace of the xy plane with the geometric requirement that no two points of the subspace lie on a vertical line drawn anywhere in the domain of the function f. It follows that a subset of points in the xy plane represents a function if and only if a vertical line (drawn anywhere in the interval representing the domain) intersects the subset at just one point. In Fig. 87, which of the subsets of

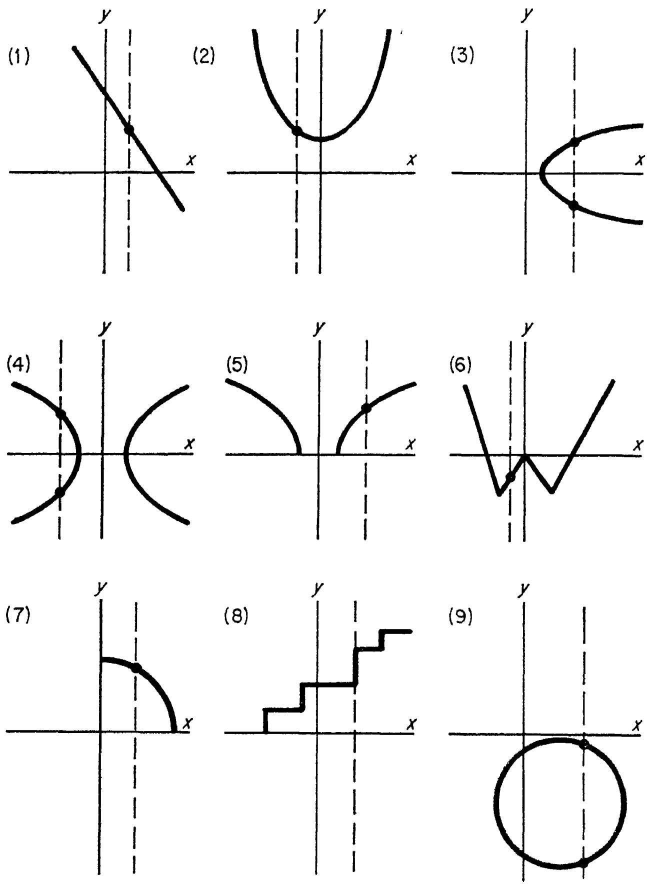


FIG. 87

the xy plane represent functions? The "vertical-line test" dictates that the subsets of (1), (2), (5), (6), and (7) represent functions, while those remaining are relations but not functions.

The graph of a function f can be used to determine whether the inverse f^{-1} is also a function. The inverse f^{-1} is a function if a horizontal line (drawn anywhere in the range of f) intersects the graph at just one point. Which of the subsets of Fig. 87 satisfy the "horizontal-line test," and what does this imply?

Example 4. In Fig. 88, the subset of points corresponding to f is a function because a vertical line intersects the graph of f at just one point for each value of its domain. The inverse f^{-1} will not be a function, since the horizontal-line test fails. Further, both g and g^{-1} will be functions, since both line tests are satisfied.

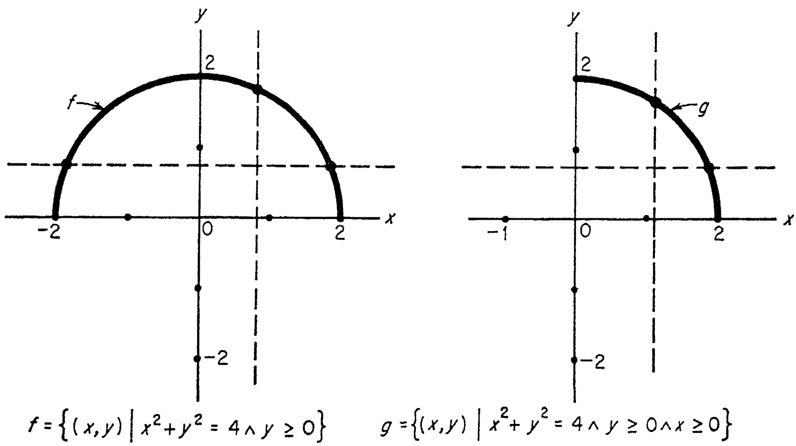


FIG. 88

4.11 COMPOSITION OF FUNCTIONS

Another way of combining two functions to form a third function is the method of composition. When two functions f and g are given, it is possible to obtain a third function h called the composite function of f on g and designated $f \circ g$. The following examples illustrate the characteristics of the composite function.

Example 1. If

$$f = \{(x,f(x)) \mid f(x) = 3x - 2\}$$

and

$$g = \{(x,g(x)) \mid g(x) = x^2\}$$

then

$$\begin{aligned} h = f \circ g &= \{(x,h(x)) \mid h(x) = f(g(x))\} \\ &= \{(x,h(x)) \mid h(x) = 3(g(x)) - 2\} \\ &= \{(x,h(x)) \mid h(x) = 3x^2 - 2\} \end{aligned}$$

If $k = g \circ f$, then

$$\begin{aligned} k = g \circ f &= \{(x,k(x)) \mid k(x) = g(f(x))\} \\ &= \{(x,k(x)) \mid k(x) = (f(x))^2\} \\ &= \{(x,k(x)) \mid k(x) = (3x - 2)^2\} \end{aligned}$$

Example 2. In general, $f \circ g \neq g \circ f$; that is, the composition of functions is noncommutative. This is apparent in Example 1. However, if f and f^{-1} are functions, then $f \circ f^{-1} = f^{-1} \circ f = i$, where

$$i = \{(x,i(x)) \mid i(x) = x\}$$

Here i is referred to as the "identity function," since all ordered pairs belonging to i are of the form (x,x). If $f = \{(x,f(x)) \mid f(x) = 2x - 1\}$, then $f^{-1} = \{(x,f^{-1}(x)) \mid f^{-1}(x) = (x + 1)/2\}$ and both are functions. Hence it follows that

$$\begin{aligned} f(f^{-1}(x)) &= 2(f^{-1}(x)) - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x \end{aligned} \qquad \text{and} \qquad \begin{aligned} f^{-1}(f(x)) &= \frac{f(x) + 1}{2} \\ &= \frac{2x - 1 + 1}{2} \\ &= x \end{aligned}$$

Here $f^{-1} \circ f = f \circ f^{-1} = i = \{(x,i(x)) \mid i(x) = x\}$.

Example 3. The most convenient way to determine the domain of a composite function $f \circ g$ is to obtain first, if possible, an expression for the value of the function at x, that is, $(f \circ g)(x)$. Those values of x for which $(f \circ g)(x)$ has meaning constitute the domain of $f \circ g$. If f and g are defined by $f(x) = 3/x - 1$ and $g(x) = x^2 - 1$, respectively, then $(f \circ g)(x) = 3/g(x) - 1$ or $(f \circ g)(x) = 3/(x^2 - 1) - 1$. Consequently, the domain of $(f \circ g)(x) = \{x \in R_e \mid |x| \neq 1\}$.

Exercise 19

$(U = R_e \times R_e)$

For the functions described by the defining conditions of Problems 1 to 6, find $f + g$, $f - g$, fg, f/g, $f \circ g$, f^{-1}, and g^{-1}.

1. $f(x) = 2x - 3$, $g(x) = 4 - x$
2. $f(x) = x^2$, $g(x) = 2x + 1$
3. $f(x) = 2x + 1$, $g(x) = x^2$
4. $f(x) = -\dfrac{4}{x}$, $g(x) = -x$
5. $f(x) = \sqrt{2 - x}$, $g(x) = 3$
6. $f(x) = \sqrt{4 - x^2}$, $g(x) = -\sqrt{4 - x^2}$

7*a*. Using f and g as defined in Example 1 of Section 4.9, show that $f + g = g + f$ and $fg = gf$. Is g/f equal to f/g? Is $f - g$ equal to $g - f$?

b. Graph $f - g$ by subtracting line segments. Graph fg and f/g by using the tabular method.

8. Let $f = \left\{(x,f(x)) \mid f(x) = \frac{2}{x}\right\}$
$g = \{(x,g(x)) \mid g(x) = x - 1\}$
$h = \{(x,h(x)) \mid h(x) = \sqrt{9 - x^2}\}$

Graph each of the following:

$f + g$	$h + g$
$f - g$	$h - g$
fg	hg
$\frac{f}{g}$	$\frac{h}{g}$

Determine the domain for $f + g$, fg, and $\frac{h}{g}$.

9. Use f, g, and h of Problem 8.
a. Graph $f \circ g$, $f \circ h$, $g \circ h$, $g \circ f$, and $h \circ f$.
b. Graph f^{-1} and g^{-1}.
c. Show that $f \circ f^{-1} = i$.
d. Show that $(f \circ g) \circ h = f \circ (g \circ h)$.
e. Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

PROJECTS

Supplementary Exercises

Given the sets $A = \{(x,y) \mid x + y = 3\}$, $B = \{(x,y) \mid 3x - 2y + 6 = 0\}$, and $C = \{(x,y) \mid y + 2 = 0\}$, interpret each set in Problems 1 to 6 graphically by shading the corresponding region.

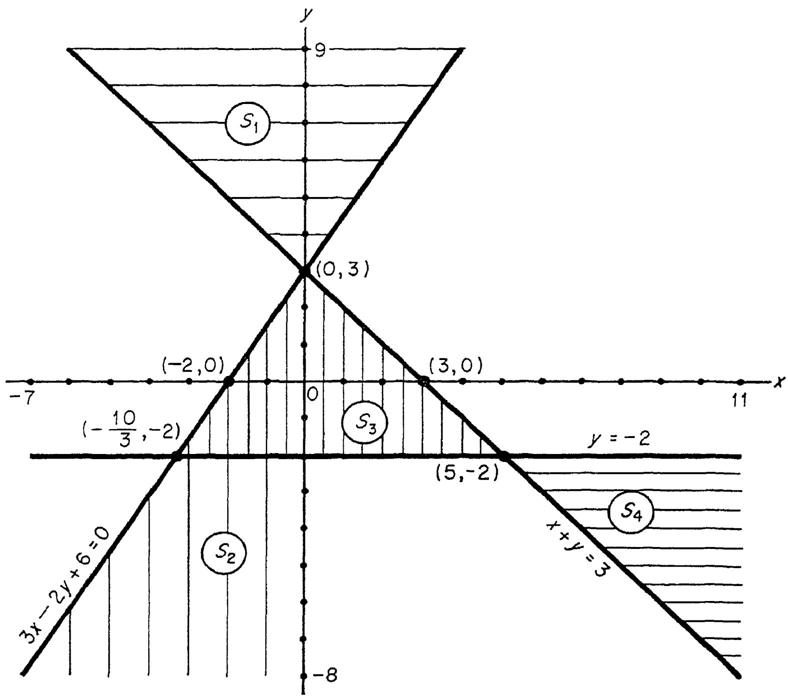
1. $F = \{(x,y) \mid x + y \leq 3 \wedge x \geq 0 \wedge y \geq 0\}$
2. $G = \{(x,y) \mid y \in [-2,0]\}$
3. $H = \{(x,y) \mid 3x - 2y + 6 \geq 0 \wedge x \leq 0 \wedge y \geq 0\}$
4. $J = \{(x,y) \mid 3x - 2y + 6 \geq 0 \wedge x + y \leq 3 \wedge y \in [-2,0]\}$
5. $K = \{(x,y) \mid 3x - 2y + 6 \geq 0 \wedge x + y \geq 3 \wedge y \geq 0\}$
6. $L = \{(x,y) \mid y + 2 \leq 0 \wedge 3x - 2y + 6 > 0 \wedge x + y \leq 3\}$

7. Give a set description for each of the shaded regions S_1, S_2, S_3, and S_4 in Fig. 89.

8. Let $A = \{(x,y) \mid x^2 + y^2 = 25\}$
$B = \{(x,y) \mid x - 2y + 5 = 0\}$
$C = \{(x,y) \mid 4x^2 = 9y\}$

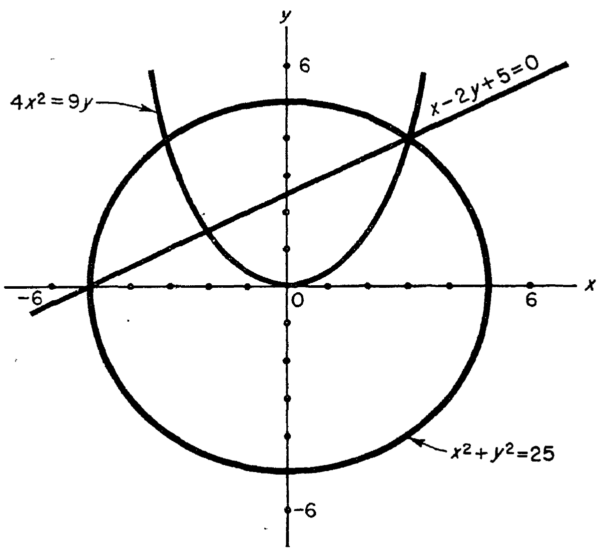
a. Find $A \cap C$, $A \cap B$, and $B \cap C$.

b. If these three sets are interpreted graphically on the same set of axes, the result is as shown in Fig. 90.



y
9
S_1
(0,3)
(−2,0)
(3,0)
x
−7
0
11
$(-\frac{10}{3},-2)$
S_3
$y=-2$
(5,−2)
S_4
S_2
$3x-2y+6=0$
$x+y=3$
−8

FIG. 89



y
6
$4x^2=9y$
$x-2y+5=0$
−6
0
6
x
$x^2+y^2=25$
−6

FIG. 90

c. Let additional set descriptions of subspaces of the xy plane be as follows:

$$D = \{(x,y) \mid x^2 + y^2 > 25\} \qquad J = \{(x,y) \mid 4x^2 - 9y < 0\}$$
$$E = \{(x,y) \mid x^2 + y^2 < 25\} \qquad K = \{(x,y) \mid y > 0\}$$
$$F = \{(x,y) \mid x - 2y + 5 > 0\} \qquad L = \{(x,y) \mid y < 0\}$$
$$G = \{(x,y) \mid x - 2y + 5 < 0\} \qquad M = \{(x,y) \mid x > 0\}$$
$$H = \{(x,y) \mid 4x^2 - 9y > 0\} \qquad N = \{(x,y) \mid x < 0\}$$

Using any of the set descriptions given in this problem and the symbols $\cap$ and $\cup$, describe each of the shaded subspaces (exclude boundary lines) shown in Fig. 91.

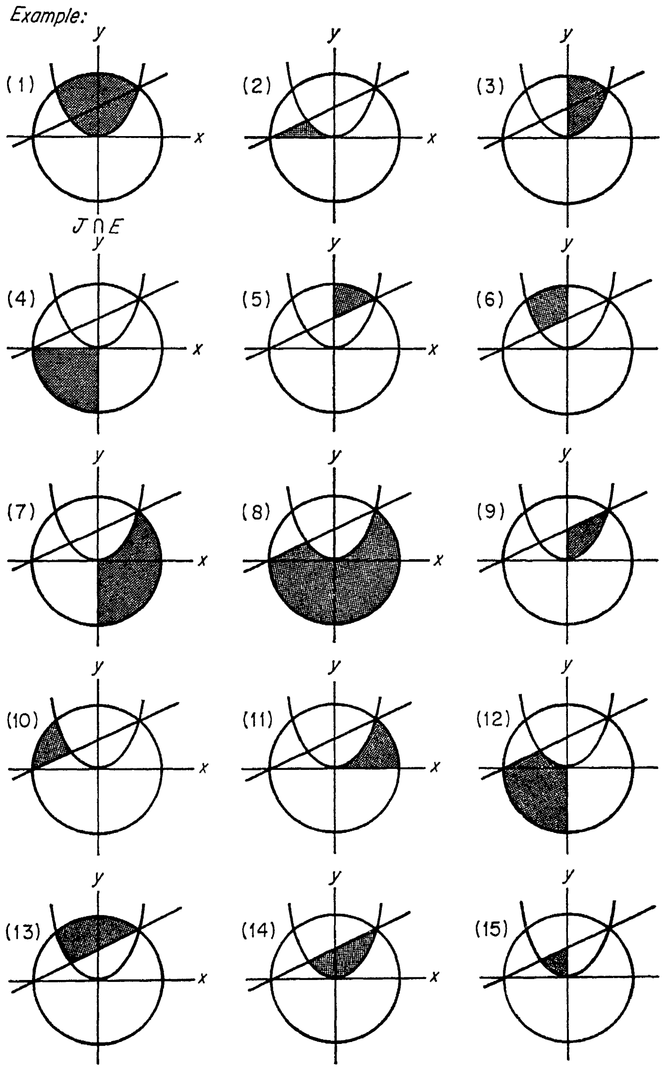


FIG. 91

9. Given that $U = R_e \times R_e$, interpret each of the following sets graphically by shading the corresponding region:

a. $A = \{(x,y) \mid x \geq 0 \land y \geq 0 \land x^2 + y^2 \leq 169\}$
b. $B = \{(x,y) \mid x \geq 0 \land y \leq 0 \land x^2 + y^2 \leq 169\}$
c. $C = \{(x,y) \mid y \geq 0 \land x^2 \leq 5y \land x^2 + y^2 \leq 169\}$
d. $D = \{(x,y) \mid x^2 + y^2 \leq 169 \land x^2 \geq 5y\}$
e. $E = \{(x,y) \mid x - y \leq 7 \land x^2 + y^2 \leq 169\}$
f. $F = \{(x,y) \mid x - 5y \geq 13 \land x - y \geq 7 \land x^2 + y^2 \leq 169\}$
g. $G = \{(x,y) \mid y \geq 0 \land x^2 + y^2 \leq 169\}$
h. $H = \{(x,y) \mid x \leq 0 \land x^2 + y^2 \leq 169\}$
i. $I = \{(x,y) \mid x + 5 \geq 0 \land x^2 \geq 5y \land y \geq 0\}$
j. $J = \{(x,y) \mid x + 5 \geq 0 \land x^2 + y^2 \geq 169 \land x^2 \leq 5y\}$

10. One of the ways that a description of subspaces in the plane may be accomplished is the use of conditions which employ inequality and absolute-value symbols. For example, if $A = \{(x,y) \mid |x - 3| < 2\}$ and $B = \{(x,y) \mid |y + 4| < 1\}$, then $A \cap B$ can be graphically interpreted as the shaded rectangle shown in Fig. 92. In

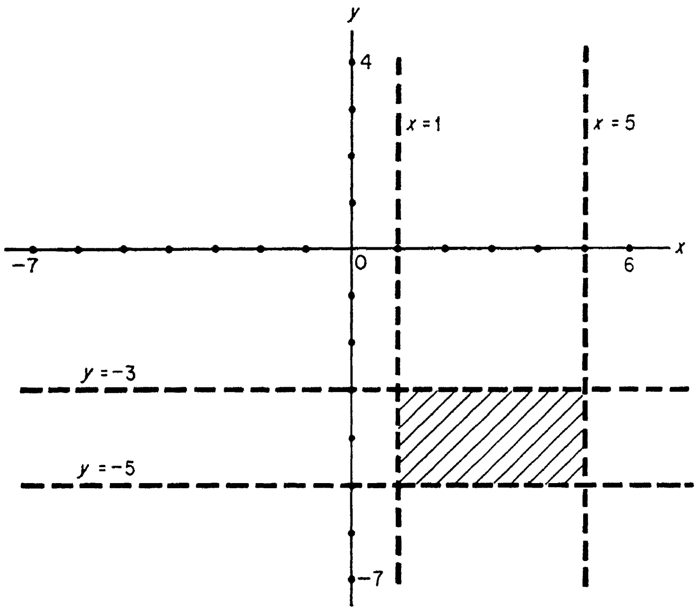


FIG. 92

order to write the conditions in sets A and B in a form that does not involve the absolute-value symbols, the following procedure may be employed:

For

$$|x - 3| < 2$$

(1) $x < 5$ when $x - 3 \geq 0$

(2) $x > 1$ when $x - 3 < 0$

Thus $x \in \,]1,5[$

For $\quad |y + 4| < 1$

(1) $\quad y < -3 \quad$ when $y + 4 \geq 0$

(2) $\quad y > -5 \quad$ when $y + 4 < 0$

Thus $\quad y \in\,]-5, -3[$

Consequently, $A \cap B = \{(x,y) \mid x \in\,]1,5[\, \wedge\, y \in\,]-5,-3[\}$, and its graphical representation is shown in Fig. 92.

Show graphically that $A \subset B$ for each of the following:

a. $A = \{(x,y) \mid |x - 3| \leq 1 \wedge |y - 1| \leq 2\}$, $B = \{(x,y) \mid x^2 + y^2 \leq 49\}$

b. $A = \{(x,y) \mid x \in [-1,1] \wedge y \in [-2,2]\}$, $B = \left\{(x,y) \,\middle|\, \dfrac{x^2}{9} + \dfrac{y^2}{16} \leq 1\right\}$

c. $A = \{(x,y) \mid |x| \leq 1 \wedge |y + 2| \leq 1\}$, $B = \{(x,y) \mid y \leq -x^2 + 2\}$

d. $A = \{(x,y) \mid y = x \wedge |y - 2| \leq 1\}$, $B = \{(x,y) \mid |x - 3| \leq 3 \wedge |y - 2| \leq 1\}$

e. $A = \left\{(x,y) \,\middle|\, \dfrac{x^2}{9} + \dfrac{y^2}{16} \leq 1\right\}$, $B = \{(x,y) \mid |x| \leq 3 \wedge |y| \leq 4\}$

11. Using one set of axes, graph the given sets and shade each region according to the directions. The final configuration could be titled "Mr. Mechanical Man."

Set	*Color*
$A = \{(x,y) \mid \|x + 1\| \leq \frac{1}{2} \wedge \|y - 9\| \leq \frac{1}{2}\}$	Black
$B = \{(x,y) \mid \|x - 1\| \leq \frac{1}{2} \wedge \|y - 9\| \leq \frac{1}{2}\}$	Black
$C = \{(x,y) \mid \|x\| \leq 1 \wedge \|y - 7\| \leq \frac{1}{2}\}$	Black
$D = \{(x,y) \mid \|x\| \leq 2 \wedge \|y - 8\| \leq 2\}$	White
$E = \{(x,y) \mid \|x\| \leq 1 \wedge \|y - 5\| \leq 1\}$	White
$F = \{(x,y) \mid \|x\| \leq 4 \wedge \|y\| \leq 4\}$	White
$G = \{(x,y) \mid \|x + 2\| \leq 1 \wedge \|y + 7\| \leq 3\}$	White
$H = \{(x,y) \mid \|x - 2\| \leq 1 \wedge \|y + 7\| \leq 3\}$	White
$J = \{(x,y) \mid \|x + 3\| \leq 2 \wedge \|y + \frac{21}{2}\| \leq \frac{1}{2}\}$	Black
$K = \{(x,y) \mid \|x - 3\| \leq 2 \wedge \|y + \frac{21}{2}\| \leq \frac{1}{2}\}$	Black
$L = \{(x,y) \mid x^2 + (y + 3)^2 \leq 1\}$	Black
$M = \{(x,y) \mid x^2 + y^2 \leq 1\}$	Black
$N = \{(x,y) \mid x^2 + (y - 3)^2 \leq 1\}$	Black
$O = \{(x,y) \mid (x - 9)^2 + (y + 1)^2 \leq 1 \wedge x \geq 9\}$	Black
$Q = \{(x,y) \mid (x + 9)^2 + (y + 1)^2 \leq 1 \wedge x \leq -9\}$	Black
$S \cap T$, where $S = \{(x,y) \mid 4x + 5y \leq 36 \wedge x \in [4,9]\}$ $T = \{(x,y) \mid 4x + 5y \geq 26 \wedge x \in [4,9]\}$	White
$V \cap W$, where $V = \{(x,y) \mid 4x - 5y \geq -36 \wedge x \in [-9,-4]\}$ $W = \{(x,y) \mid 4x - 5y \leq -26 \wedge x \in [-9,-4]\}$	White
$Z = \{(x,y) \mid \|x\| \leq \frac{1}{4} \wedge \|y - 8\| \leq \frac{1}{4}\}$	Black

12. The set of complex numbers C is an extension of the set of real numbers R_e so as to include numbers involving the unit $i = \sqrt{-1}$. It can be shown that the solution set of an equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ will be nonempty and will contain complex numbers as elements if the coefficients $a,b,c \in C$.

Accordingly, the intersection of two sets such as $A = \{(x,y) \mid x^2 + y^2 = 9\}$ and $B = \{(x,y) \mid y = 5\}$ will be nonempty if U is chosen as $C \times C$. $A \cap B$ is obtained

in the following manner:

$$x^2 + y^2 = 9 \wedge y = 5$$
$$x^2 + 25 = 9$$
$$x^2 + 16 = 0$$
$$x^2 - 16i^2 = 0 \qquad \text{since } i^2 = -1$$
$$(x - 4i)(x + 4i) = 0$$

$$x = 4i \qquad\qquad x = -4i$$
$$y = 5 \qquad\qquad y = 5$$

Hence

$$A \cap B = \{(-4i,5),(4i,5)\}$$

The graph of sets A and B in the xy plane furnishes additional evidence that the ordered pairs of $A \cap B$ are not ordered pairs of real numbers, since $A \cap B = \phi$ if $U = R_e \times R_e$.

If $x \in C$, tabulate the solution set of each of the following equations:

a. $x^2 + 4 = 0$ *b.* $x^2 + 3x + 4 = 0$ *c.* $4x^2 - 3x + 2 = 0$
d. $x^2 + x + 1 = 0$ *e.* $3x(x - 1) + 1 = 0$

If $(x,y) \in C \times C$, find $A \cap B$ for each of the following:

f. $A = \{(x,y) \mid x^2 + 2y^2 = 3\}$
$B = \{(x,y) \mid x - y = 3\}$

g. $A = \{(x,y) \mid xy = 4\}$
$B = \{(x,y) \mid x + y = 1\}$

h. $A = \{(x,y) \mid x^2 - 4y = 0\}$
$B = \{(x,y) \mid x - 2y = 1\}$

i. $A = \{(x,y) \mid 5x^2 + 2y^2 = 13\}$
$B = \{(x,y) \mid 4x^2 - 7y^2 = 19\}$

j. $A = \{(x,y) \mid 8x^2 + 9y^2 = 7\}$
$B = \{(x,y) \mid 4x^2 - 2y^2 = -3\}$

13. In the physical sciences frequent contact is made with sets of ordered pairs, such as

$$A = \{(2,6),(3,9),(4,12),(5,15),(6,18), \ldots\}$$

or

$$B = \{(4,1),(8,2),(3,\tfrac{3}{4}),(5,\tfrac{5}{4}), \ldots\}$$

In set A the components of each ordered pair are related in such a way that the ratio of the second component to the first component is the constant $\frac{1}{3}$; and in set B, the ratio is the constant $\frac{1}{4}$.

Whenever a physical situation produces ordered pairs that are related in this described manner, then the components of the ordered pairs are said to vary directly. Symbolically, $f = \{(x,y) \mid y/x = k\} = \{(x,y) \mid y = kx\}$ (where $k \neq 0$ is a constant) implies in words that "y varies directly as x" or "y is directly proportional to x." In such situations the requirements of a function are satisfied, where its domain is specified or understood implicitly so as to conform to the physical phenomena under examination.

If each of the ordered pairs of a given set is so related that the product of its components is a constant, then the components of the ordered pairs are said to vary inversely. Symbolically, $g = \{(x,y) \mid xy = k\} = \{(x,y) \mid y = k/x\}$ (where $k \neq 0$ is a constant) implies in words that "y varies inversely as x" or "y is inversely proportional to x." For example, the ordered pairs $(3,1)$, $(6,\frac{1}{2})$, and $(4,\frac{3}{4})$ are elements of the set $\{(x,y) \mid xy = 3\}$ and satisfy the definition of inverse variation.

Assuming that each of the sets in a to e contains ordered pairs all of which have been formed in the same way, determine whether a direct or an inverse variation exists and describe the set in the form $\{(x,y) \mid P_{xy}\}$.

a. $\{(4,6),(2,3),(5,\frac{15}{2}), \ldots\}$.
b. $\{(5,1),(10,2),(15,3), \ldots\}$.
c. $\{(5,\frac{1}{3}),(\frac{2}{3},\frac{5}{2}),(\frac{1}{6},10), \ldots\}$.
d. $\{2,\frac{9}{20}),(\frac{1}{5},\frac{9}{2}),(\frac{1}{6},\frac{27}{5}), \ldots\}$.
e. $\{(\frac{1}{3},\frac{1}{8}),(\frac{1}{4},\frac{3}{32}),(\frac{3}{7},\frac{9}{56}), \ldots\}$.
f. If $f = \{(x,y) \mid y = 3x\}$, graph f, f^{-1}, and $f + f^{-1}$.
g. If $g = \{(x,y) \mid xy = 5\}$, graph g, g^{-1}, and $g + g^{-1}$.

14. As a final problem in this project list, it is suggested that a careful study be made of the ordered pairs connected with the unit circle. This offers an excellent opportunity for reviewing set concepts, set symbolisms, relations, and functions. The following hints of procedure are included as helpful suggestions.

a. Let $U = \{(x,y) \in R_e \times R_e \mid x^2 + y^2 = 1\}$. If U, the universe in this discussion, is graphed, then the vertical-line test confirms that U is a relation and not a function. Furthermore, the domain and range of U are both equal to $[-1,1]$. The graph of U is referred to as the "unit circle," as shown in Fig. 93.

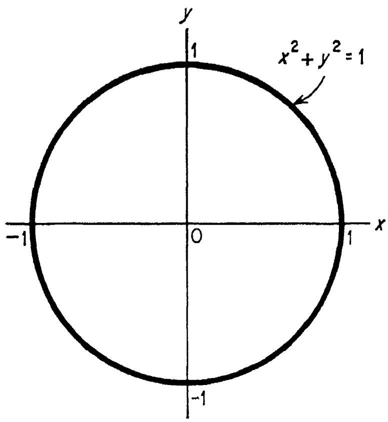


FIG. 93

b. Let $U_1 = \{(1,0),(-1,0)\}$ and $U_2 = \{(0,1),(0,-1)\}$, as shown in Fig. 94. Interpret, graph, determine whether the relation is a function, and specify the domain and range in each of the following: $U_1 \cup U_2$; $(U_1)'$; $U_3 = (U_1 \cup U_2)' = (U_1)' \cap (U_2)'$; $U_4 = U \cap (U_1)'$; $U_5 = U \cap (U_2)'$; $U_1 \cap U_2$; $(U_1 \cap U_2)'$; $U \cap (U_1 \cap U_2)'$; $(U_5)'$; $U \cap (U_1 \cap U_2) = U \cap U_1 \cap U_2$.

c. An examination of sets such as $U_6 = \{(x,y) \in U \mid x \in]0,1[\wedge y \in]0,1[\}$ and $U_7 = \{(x,y) \mid x \in]-1,0[\wedge y \in]0,1[\}$ exposes ordered pairs associated with those points on the circumference of the unit circle in quadrants I and II, respectively. Note that these set descriptions exclude the points $(1,0)$, $(-1,0)$, and $(0,1)$, as shown in Fig. 95. The description of the sets corresponding to the points on the unit circle belonging to quadrants III and IV is left as an exercise.

d. The review of special types of simultaneous systems occurring in algebra may be accomplished by the following type of problem. Let $A = \{(x,y) \mid x^2 + y^2 = 1 \wedge x + 2y = 2\}$. This set can also be described as $\{(x,y) \in U \mid x + 2y = 2\}$, since

$(x,y) \in U$ implies that one of the equations in the simultaneous system is automatically $x^2 + y^2 = 1$. If set A is interpreted graphically, the solution set corresponding to the points of intersection is $\{(0,1),(\frac{4}{5},\frac{3}{5})\}$. This material may be extended to include simultaneous systems which lead to the empty set as well as solution sets which contain either one or more ordered pairs belonging to U.

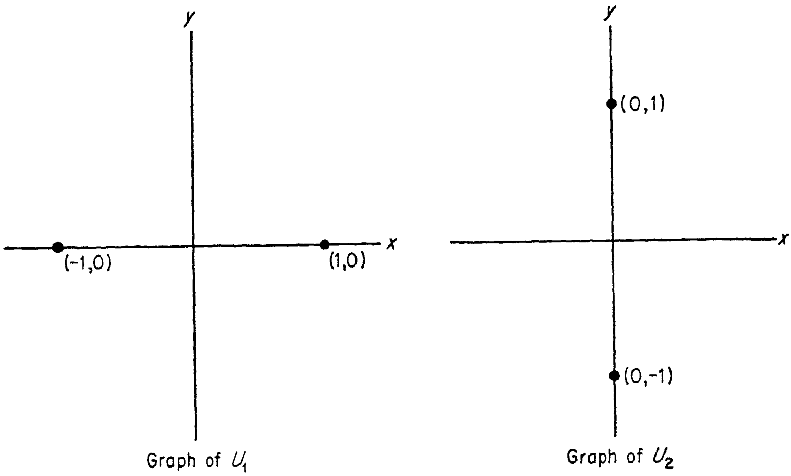


FIG. 94

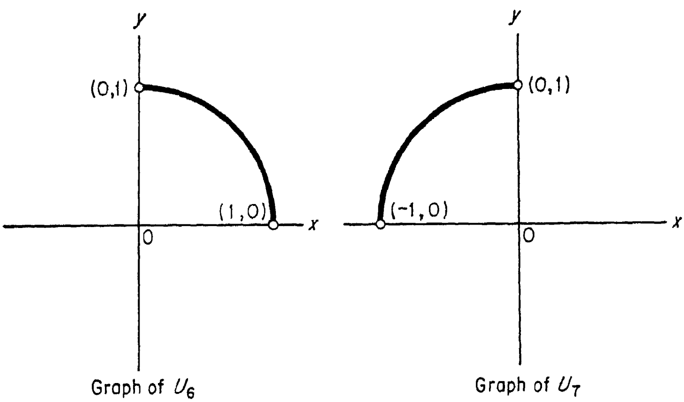


FIG. 95

e. Number identities can be created which effectively review techniques in arithmetic. A problem of the following type illustrates how this may be accomplished. If, for example, ordered pairs are selected in U_6 and U_7 as described in part *c*, it can be verified that $(1 - x)/y$ is always equal to $y/(1 + x)$. In U_6 and U_7, $(1 + x)y \neq 0$ and thus $(1 - x)/y = y/(1 + x)$ is equivalent to $x^2 + y^2 = 1$. However, the statement $(1 - x)/y = y/(1 + x)$ is not true for all $(x,y) \in U$, since division by 0 is not possible. Is $(1 - x)/y = y/(1 + x)$ true for all elements $(x,y) \in U_3$, where $U_3 = (U_1 \cup U_2)'$? Are there other elements in $(U_3)'$ for which this statement is true? Problems of this

type add new flavor and provide exercises which involve fractions, number substitutions for letters, and other features sought in arithmetic and algebraic techniques. Number identities can be created from any known trigonometric identity by replacing the trigonometric expressions in the identity with ratios formed from the x and y of the ordered pairs (x,y) belonging to U. However, care must be taken to define the particular subset of U which becomes involved in the verification of the identity.

f. Every $(x,y) \in U_3$ where $U_3 = (U_1)' \cap (U_2)'$ has six important associated numbers called ratios, namely, x, y, $\frac{y}{x}$, $\frac{x}{y}$, $\frac{1}{x}$, $\frac{1}{y}$. Every $(x,y) \in U_1$ has only four important associated numbers, namely x, $y = 0$, $y/x = 0$, and $1/x$, since $x/y = x/0$ (nonexistent) and $1/y = \frac{1}{0}$ (nonexistent). Similarly, every $(x,y) \in U_2$ has four important associated numbers, namely, $x = 0$, y, $x/y = 0$, and $1/y$, since $y/x = y/0$ (nonexistent) and $1/x = \frac{1}{0}$ (nonexistent). This material suggests problems of the following type:

(1) Given $y/x = \frac{2}{3}$, find the associated ordered pair or pairs of U, and the remaining ratios (see part *d*).

(2) Given $y/x = \frac{1}{4}$ and $x < 0$, find the associated ordered pair or pairs of U, and the remaining ratios. Many exercises which involve ratio and proportion emanate from the ratios associated with each (x,y) belonging to U.

The ideas of this project are preparatory work for trigonometry. If further enrichment is desired, one may proceed into formal trigonometry by establishing an association between each $(x,y) \in U$, an arc measure, and the measure of the angle subtended by this arc on the unit circle.

5

Mathematical Structures

5.1 INTRODUCTION

The mathematician is being challenged to make his discipline available to all areas of learning. To meet this responsibility he has introduced the concept of a mathematical structure or system, whereby a framework is established for which he hopes to find analogues that apply to the physical universe. The idea of such a structure comes into being by first accepting various notions on an intuitive basis. The structure itself is a set of abstract elements, undefined terms, and various rules. The investigation of what can be discovered in the way of useful information determines the effectiveness of the invented mathematical structure. The purpose of this chapter is to introduce notions that will exhibit the internal characteristics and properties of particular mathematical structures known as Boolean algebras, groups, and fields.

5.2 CONCEPT OF A MATHEMATICAL STRUCTURE

Every mathematical structure or system has its origin in various undefined terms, such as set, number, and point. Since concepts are defined on the basis of other concepts and terms, there must exist a small core of words representing the basic vocabulary for defining newer terms and concepts which arise in the development of a mathematical structure. This initial core of terms or concepts is originally undefined and is accepted intuitively.

Given a nonempty set of elements U, certain relations can be introduced between elements. A definition of equality is essential so as to distinguish one element from another. If $x = y$, then x and y are identical, or the different symbols x and y represent the same element. These symbols may be substituted for each other in any mathematical expression.

A mathematical structure will contain binary operations defined on U, that is, ways of combining two elements $\in U$. If an operation combines two elements $\in U$ to produce a unique element $\in U$, then the operation is referred to as a closed operation. For example, the set of

integers is closed under addition and multiplication but is not closed under the operation of division. Hence, a set U is closed under the binary operation $*$ if each ordered pair $\in U \times U$ can be associated with a unique element $\in U$; or in other words if $a,b \in U$ with operation $*$, then $a * b \in U$. Further examples of such operations are the addition and multiplication of rational numbers, union and intersection of sets, etc.

Since one or more operations are basic ingredients of a mathematical structure, various properties of each such operation must be examined with respect to a particular set of elements. In general, if $a,b \in U$ with operation $*$, then

a. $*$ is closed if $a * b \in U$.
b. $*$ is commutative if $a * b = b * a$.
c. $*$ is associative if $(a * b) * c = a * (b * c)$.

For example, if $U = N$, I, F, or R_e and $*$ is either $+$ or $\cdot$, then these properties are possessed by the resulting mathematical structures.

Example 1. Let $a * b$ be defined as the average of two numbers, $a * b = (a + b)/2$, where $a \in F$ and $b \in F$. Does this operation possess the properties of closure, commutativity, and associativity?

Closure: F is closed under the operation $*$, since the average of any two rational numbers is also a rational number.

Commutativity: $a * b = \dfrac{a+b}{2}$ and $b * a = \dfrac{b+a}{2} = \dfrac{a+b}{2}$, since the addition of rational numbers is already known to be commutative. Hence this property holds for the operation $*$.

Associativity:

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c = \frac{(a+b)/2 + c}{2} = \frac{a+b+2c}{4}$$

$$a * (b * c) = a * \left(\frac{b+c}{2}\right) = \frac{a + (b+c)/2}{2} = \frac{2a+b+c}{4}$$

Since $\dfrac{a+b+2c}{4} \neq \dfrac{2a+b+c}{4}$, the operation $*$ does not possess the associative property.

These conclusions can be tested numerically by replacing a, b, and c with specific rational numbers. It is important to note that the existence of commutativity for a defined operation does not ensure associativity for this same operation.

Example 2. In each of the illustrations that have been considered, the operation $*$ was defined with respect to an infinite set of elements. However, the operation $*$ can be defined for a finite set of elements as well. For example, the operation $*$ defined on the set $U = \{e,o\}$ may be described by the following table:

$*$	e	o
e	e	o
o	o	e

where
$$\begin{aligned} e * e &= e \\ e * o &= o \\ o * e &= o \\ o * o &= e \end{aligned}$$

To find $e * o$, use row 1 and column 2. The result o is located in the corresponding cell. Since every cell in the table is filled by an element $\in U$, the operation $*$ is closed. Further, the operation $*$ is commutative, since a study of the table exposes a symmetry about the diagonal, moving from upper left to lower right. To test for associativity, all the possible eight triples of three elements must be examined.

For example, $(e * o) * o = e * (o * o)$. Using the table and examining each member separately, we find

$$\begin{aligned} (e * o) * o &= o * o \\ &= e \end{aligned} \qquad \begin{aligned} e * (o * o) &= e * e \\ &= e \end{aligned}$$

Accordingly, a complete examination will reaffirm that the operation $*$ with respect to $U = \{e,o\}$ possesses the properties of closure, commutativity, and associativity. If elements e and o represent "even" and "odd" integers, respectively, and $*$ represents the binary operation of addition, then a simple interpretation of this structure for sums of even and odd integers becomes apparent.

The most important component of any mathematical structure is its set of postulates or assumptions. If the basic assumptions of the system are not contradictory, then the set of postulates is said to be consistent. The usual intent when developing a mathematical structure is to hold the number of undefined terms and assumptions to a minimum. However, for pedagogical purposes certain properties are included as basic postulates even though they may be logical consequences of previous postulates. Still another significant characteristic of a mathematical structure is a set of definitions which represent agreements concerning symbolism and terminology so that newer concepts, properties, and theorems can be derived.

To summarize, a mathematical structure or system consists of a set of elements, operations, relations, postulates, theorems, and definitions. If A represents a set of elements, and $*$ an operation, then $\{A: *\}$ will symbolize a mathematical structure. The set of elements is clearly characterized or initially described by a set of statements, called postulates. The postulates represent the "rules" or "laws" of the system and govern the meaning of the symbols that are used to represent the elements, relations, and operations of the system. Other statements, called theorems, are formed and proved as a consequence of the original set of postulates and accepted rules of logic. When these theorems are proved, they possess the same validity in the system as the original set of postulates, since each of these theorems results as a logical consequence of the postulates.

It is possible to construct many different mathematical systems depending on the choice of different sets of elements, relations, operations, and postulates. One particular system is not necessarily better than another; each one is studied on the basis of its own merits and interpretations. Mathematical systems are often developed with definite interpretations in mind, but frequently a mathematical system is devised in which no physical interpretation of the terms and symbols is considered. After a particular abstract system has been constructed, it may be interpreted in many different ways. If all the postulates of the system are true for a specific interpretation of the terms and symbols of the system, then this specific interpretation represents a "model" of it. The applied scientist studies a particular abstract system so as to fit it to some aspect of the physical universe. Sometimes he succeeds and other times he fails, not because the mathematical system is incorrect but because the physical situation is not a correct model or interpretation of the system under consideration. Mathematical systems may be looked upon as games that involve certain objects (elements) and are played according to specified rules (postulates).

Exercise 20

1. In each of the following, determine whether the indicated operation is closed with respect to the given set. Give a counterexample for each operation that does not possess the closure property.

	Set	*Operation*
a.	$p,q \in \{\text{integers}\}$	Subtraction
b.	$p,q \in \{\text{even integers}\}$	Addition
c.	$p,q \in \{\text{even integers}\}$	Multiplication
d.	$p,q \in \{\text{odd integers}\}$	Subtraction
e.	$p,q \in \{\text{odd integers}\}$	Multiplication
f.	$p,q \in \{\text{odd integers}\}$	Addition
g.	$p \in \{\text{odd integers}\}$	Forming the square of p

	Set	*Operation*
h.	$p,q \in$ {primes}	Addition
i.	$p,q \in$ {primes}	Multiplication
j.	$p \in$ {positive integers}	Forming the reciprocal of p
k.	$p \in$ {integers}	Forming the negative of p
l.	$p \in$ {rational numbers}	Forming the reciprocal of p
m.	$p \in$ {rational numbers}	Finding a square root of p
n.	$p \in$ {integers}	Finding a square root of p

2. The operation $*$ is defined as indicated in each of the following problems. Determine for each case whether the operation $*$ is closed, commutative, and associative with respect to the designated set. Illustrate with a counterexample those situations where a property does not hold.

a. $a,b \in F$ and $a * b = \dfrac{a+b}{3}$

b. $a,b \in I$ and $a * b = a + 2b$

c. $a,b \in I$ and $a * b = a$

d. $a,b \in I$ and $a * b = \dfrac{ab}{2}$

Examples. $3 * 4 = 3$, $(-5) * (-2) = -5$, $(-2) * (-5) = -2$

e. $a,b \in I$ and $a * b = b$

f. $a,b \in F$ and $a * b = a - b$

g. $a,b \in N$ and $a * b$ means "take the smallest number"

Examples. $3 * 5 = 3$, $2 * 9 = 2$, $9 * 2 = 2$, $4 * 4 = 4$

h. $a,b \in N$ and $a * b$ means "take the largest number"
i. $a,b \in I$ and $a * b = ab + a - b$ (note that $b * a = ba + b - a$)
j. $a,b \in I$ and $a * b = a + b - ab$

3. Determine whether the three properties mentioned in Problem 2 hold for the operation $*$ as defined with respect to the designated set.

a. $U = \{e,o\}$

$*$	e	o
e	e	e
o	e	o

b. $U = \{1,-1,i,-i\}$

$*$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	-1
$-i$	$-i$	i	-1	1

Test the associative law for the following cases:

$$(i * i) * i = i * (i * i)$$
$$(-1 * i) * i = -1 * (i * i)$$
$$[(-i) * i] * (-i) = -i * [i * (-i)]$$
$$[(-i) * (-i)] * 1 = -i * [(-i) * 1]$$

c. $U = \{a,b,c\}$

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

d $U = \{a,b,c\}$

$*$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

e. $U = \{a,b,c\}$

$*$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

f. $U = \{0,1,2,3,4\}$

$*$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Test a few cases for associativity.

5.3 THE MATHEMATICAL SYSTEM CALLED AN "ALGEBRA OF SETS"

The "algebra of sets" is an example of a mathematical system. The algebra of sets has a collection of sets, including $\emptyset$ and U (null set and universe), as its elements; "$\subseteq$" and "$=$" as its relations; two binary operations represented by $\cup$ and $\cap$; and the concept of complementation represented by $'$.

If S is a collection of subsets of a universal set U, and if for every A and B in S, $A \cup B$, $A \cap B$, and A' are also elements of S, then S is called an "algebra of sets," provided that the definitions and Laws 1 to 9, which follow, are satisfied.

Definitions:

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$
$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$
$$A' = \{x \in U \mid x \notin A\}$$

If S is an algebra of sets and if $A, B, C, \ldots, \emptyset, U, \ldots$ are elements of S, then the following hold for $\cup$, $\cap$, and $'$.

Identity Laws

1*a*. $A \cup \emptyset = A$ 1*b*. $A \cap \emptyset = \emptyset$

2*a*. $A \cup U = U$ 2*b*. $A \cap U = A$

Idempotent Laws

3*a*. $A \cup A = A$ 3*b*. $A \cap A = A$

Complement Laws

4*a*. $A \cup A' = U$ 4*b*. $A \cap A' = \emptyset$
5*a*. $(A')' = A$ 5*b*. $\emptyset' = U$; $U' = \emptyset$

Commutative Laws

6*a*. $A \cup B = B \cup A$ 6*b*. $A \cap B = B \cap A$

Associative Laws

7*a*. $(A \cup B) \cup C = A \cup (B \cup C)$
7*b*. $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws

8*a*. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8*b*. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Laws

9*a*. $(A \cup B)' = A' \cap B'$ 9*b*. $(A \cap B)' = A' \cup B'$

The mathematical system called the real-number system has certain properties analogous to those of the algebra of sets. The operations of addition "$+$" and multiplication "$\cdot$" correspond to the operations of union "$\cup$" and intersection "$\cap$." In both systems rules are devised for these operations; i.e., some rule or procedure assigns to each pair of elements some unique element from the associated universe. Frequently, "$\cup$" is called "logical addition" and "$A \cup B$" is the "logical sum" of sets A and B. Similarly, "$A \cap B$" is called the "logical product" of sets A and B and "$\cap$" is referred to as "logical multiplication."

The algebra of real numbers and the algebra of sets are alike in certain respects and unlike in others. Some of these similarities and differences will now be noted.

a. The elements of the real-number system (i.e., real numbers) obey the commutative and associative laws for addition and multiplication, but only one of the distributive laws, namely, 8*b* and not 8*a* (see Section 2.7). In the real-number system, multiplication is distributive over addition, but addition is not distributive over multiplication; i.e., $3 \cdot (4 + 5) = 3 \cdot 4 + 3 \cdot 5$, but $3 + (4 \cdot 5) \neq (3 + 4) \cdot (3 + 5)$. In the algebra of sets, union is distributive with respect to intersection, and intersection is distributive with respect to union, as indicated in Laws 8*a* and 8*b*.

b. Another property shared by both systems is the existence of identity elements. In the real-number system, 0 (zero) is the additive identity and 1 (one) is the multiplicative identity; i.e., when 0 is added to any real number a the sum is that same number a, and when 1 is multi-

plied by any real number a the product is that same real number a. Symbolically,

(1) If $a \in R_e$, then $a + 0 = a$.

(2) If $a \in R_e$, then $a \cdot 1 = a$.

Correspondingly, the algebra of sets has a unique identity element "$\emptyset$" for union and a unique identity element "U" for intersection; i.e., the union of $\emptyset$ with any element A of S yields A, and the intersection of U with A also yields A.

c. The idempotent laws for union and intersection state that the union or intersection of a set with itself yields the same set. For real numbers this is not generally true, since $2 + 2 \neq 2$ and $3 + 3 \neq 3$, though $0 + 0 = 0$. Similar illustrations will indicate that the analogy does not hold for the multiplication of real numbers.

d. It is of interest to note that every real number has a unique additive inverse; i.e., the addition of each real number a to its inverse $(-a)$ yields 0, the additive identity. Also every real number a, except zero, has a unique multiplicative inverse a^{-1}; i.e., the multiplication of each real number by its inverse yields 1, the multiplicative identity. A like situation does not hold for the algebra of sets. The element A of S has no corresponding element such that its union with the set A will yield the identity element $\emptyset$ (excluding the case $\emptyset \cup \emptyset = \emptyset$). With the exception $U \cap U = U$, this is also true for the operation of intersection. For example, if A = a finite set of real numbers, then

$$A' = \{\text{all the other real numbers}\}$$

and $A \cup A' = R_e$, while $A \cap A' = \emptyset$.

5.4 THE MEMBERSHIP METHOD AND VENN DIAGRAMS

For each element $A \in S$, there is associated another element A', called the complement of A. This complement A' has the following properties:

$$A \cup A' = U \qquad A \cap A' = \emptyset \qquad (A')' = A$$

The complementary set also obeys De Morgan's laws. In words, Law 9*a* states that the complement of a union of two sets is the intersection of their complements, while Law 9*b* states that the complement of an intersection of two sets is the union of their complements. The proof of any law for the algebra of sets can be accomplished through the use of membership tables. For example, the proof of Law 9*b*, namely, $(A \cap B)' = A' \cup B'$, proceeds as follows.

Example 1. Consider any element a of U. In terms of this element and the two sets A and B, four distinct possibilities are listed. The element a can:

1. Belong to A and also belong to B
2. Belong to A but not to B
3. Belong to B but not to A
4. Not belong to A and not belong to B.

These four possibilities appear in the first two columns of Table 1. The other columns are developed in successive order so as to verify the given law.

Table 1

	(1) A	(2) B	(3) A'	(4) B'	(5) $A \cup B$	(6) $(A \cup B)'$	(7) $A' \cap B'$
1.	$\in$	$\in$	$\notin$	$\notin$	$\in$	$\notin$	$\notin$
2.	$\in$	$\notin$	$\notin$	$\in$	$\in$	$\notin$	$\notin$
3.	$\notin$	$\in$	$\in$	$\notin$	$\in$	$\notin$	$\notin$
4.	$\notin$	$\notin$	$\in$	$\in$	$\notin$	$\in$	$\in$

Row 3 illustrates the possibility that $a \notin A$ but $a \in B$. Hence in columns 3 and 4, $a \in A'$ and $a \notin B'$. By use of columns 1 and 2 column 5 is formed, since if $a \notin A$ and $a \in B$, then $a \in A \cup B$. Since $a \in A \cup B$, it follows that $a \notin (A \cup B)'$. Column 7 is formed from columns 3 and 4 in a like manner. Accordingly, $(A \cup B)' = A' \cap B'$ is based on the fact that the last two columns are identical; i.e., when a is an element of $(A \cup B)'$, it is also an element of $A' \cap B'$; and when $a \notin (A \cup B)'$, then $a \notin A' \cap B'$. This law can be verified by using a Venn diagram (Fig. 96) and examining corresponding regions.

$(A \cup B)' = A' \cap B'$

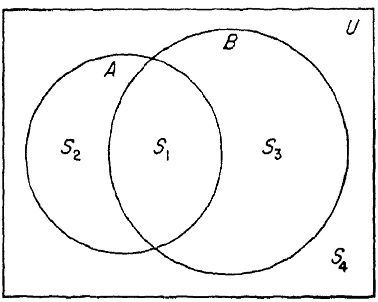


FIG. 96

Set	A	B	A'	B'	$A \cup B$	$(A \cup B)'$	$A' \cap B'$
Regions	S_1,S_2	S_1,S_3	S_3,S_4	S_2,S_4	S_1,S_2,S_3	S_4	S_4

Thus $(A \cup B)' = A' \cap B'$, since both sets represent the same region.

Example 2. It is of interest to examine by either the membership method or Venn diagrams all the basic laws for the algebra of sets. As

a second example, the proof of Law 8b, namely,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

is given in the membership table (Table 2), which includes eight possibilities.

Table 2

A	B	C	$B \cup C$	$A \cap B$	$A \cap C$	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
$\in$	$\in$	$\notin$	$\in$	$\in$	$\notin$	$\in$	$\in$
$\in$	$\notin$	$\in$	$\in$	$\notin$	$\in$	$\in$	$\in$
$\in$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\in$	$\in$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\in$	$\notin$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\notin$	$\in$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$

ties. The Venn diagram takes the form shown in Fig. 97, and the regions are given in Table 3.

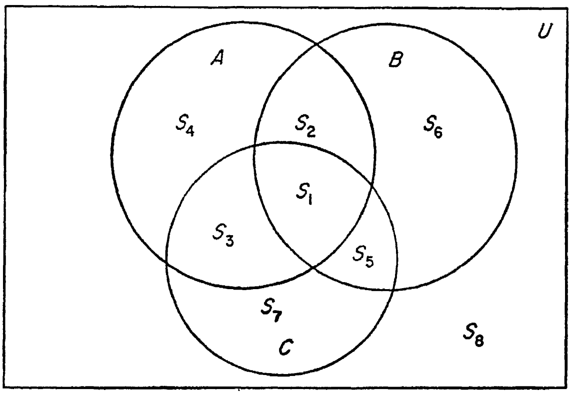


FIG. 97

Table 3

A	B	C	$B \cup C$	$A \cap B$	$A \cap C$	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
S_1	S_1	S_1	S_1	S_1	S_1	S_1	S_1
S_2	S_2		S_2	S_2		S_2	S_2
S_3		S_3	S_3		S_3	S_3	S_3
S_4							
	S_5	S_5	S_5				
	S_6		S_6				
		S_7	S_7				

5.5 PROOF OF THEOREMS BY USE OF LAWS AND DEFINITIONS

The following examples illustrate how Laws 1 to 9, Section 5.3, may be used in carrying out proofs of theorems.

Example 1. Prove that if A and B are subsets of U, then:

a. $A = (A \cap B) \cup (A \cap B')$ *b.* $(A \cap B') \cup B = A \cup B$

c. $A \cap \emptyset = \emptyset$ *d.* $(A \cap B') \cap A' = \emptyset$

a. $A = (A \cap B) \cup (A \cap B')$

Proof:	*Authority*
$(A \cap B) \cup (A \cap B') = A \cap (B \cup B')$	Law 8*b*
$= A \cap U$	Law 4*a*
$= A$	Law 2*b*

If the steps are reversed, then:

$A = A \cap U$	Law 2*b*
$= A \cap (B \cup B')$	Law 4*a*
$= (A \cap B) \cup (A \cap B')$	Law 8*b*

b. $(A \cap B') \cup B = A \cup B$

Proof:	*Authority*
$(A \cap B') \cup B = B \cup (A \cap B')$	Law 6*a*
$= (A \cup B) \cap (B' \cup B)$	Laws 8*a* and 6*a*
$= (A \cup B) \cap U$	Law 4*a*
$= A \cup B$	Law 2*b*

c. $A \cap \emptyset = \emptyset$

Proof:	*Authority*
$\emptyset = A \cap A'$	Law 4*b*
$= A \cap (A' \cup \emptyset)$	Law 1*a*
$= (A \cap A') \cup (A \cap \emptyset)$	Law 8*b*
$= \emptyset \cup (A \cap \emptyset)$	Law 4*b*
$= (A \cap \emptyset) \cup \emptyset$	Law 6*a*
$= A \cap \emptyset$	Law 1*a*

The proof of part *d* is left as an exercise.

Example 2. The definitions for subset, union, intersection, and complementation are involved in the proofs of the following theorems:

a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

b. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

c. $A \subseteq B$ if and only if $A \cup B = B$.

d. $B \subseteq A$ if and only if $A \cap B = B$.

e. If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

f. $A \subseteq B$ if and only if $B' \subseteq A'$.

These theorems may be illustrated by Venn diagrams but shall be proved here more formally.

a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let x represent any arbitrary element of A. Since $A \subseteq B$, then by the definition of a subset, $x \in B$. Likewise, since $B \subseteq C$ it follows that $x \in C$. Hence, $A \subseteq C$, since every element of A is also an element of C.

b. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Since $A \subseteq B$, every element of A is also an element of B. Since $B \subseteq A$, every element of B is also an element of A. Hence every element of A is an element of B and every element of B is an element of A; therefore $A = B$, by the definition of set equality.

d. $B \subseteq A$ if and only if $A \cap B = B$.

Proof:

(1) If $A \cap B = B$, then $B \subseteq A$.

Let x represent any arbitrary element of $A \cap B$. Then if $x \in A \cap B$ it follows that $x \in A$ and $x \in B$. Thus $A \cap B \subseteq A$ and since it is given that $A \cap B = B$ then $B \subseteq A$.

(2) If $B \subseteq A$, then $A \cap B = B$.

Let x represent any arbitrary element of B. Then by the definition of subset, $x \in A$. Since $x \in B$ and $x \in A$, it follows that $x \in A \cap B$. Thus $B \subseteq A \cap B$. If $x \in A \cap B$, then $x \in B$ and $A \cap B \subseteq B$. Consequently, $A \cap B = B$ because $A \cap B \subseteq B$ and $B \subseteq A \cap B$.

f. $A \subseteq B$ if and only if $B' \subseteq A'$.

Proof:

(1) If $A \subseteq B$, then $B' \subseteq A'$.

Let x represent any element in B'; then $x \notin B$, by the definition of complement. If $x \notin B$, then $x \notin A$, since, from $A \subseteq B$, every element in A must be contained in B. Therefore $x \in A'$ and since $x \in B'$ and $x \in A'$, it follows that $B' \subseteq A'$.

(2) If $B' \subseteq A'$, then $A \subseteq B$.

Let x represent any element of A; then $x \notin A'$ and $x \notin B'$, from $B' \subseteq A'$. Hence $x \in B$ and $A \subseteq B$. Part 2 can be proved by using part 1; that is, if $A \subseteq B$, then $(B)' \subseteq (A)'$. Replacing A by B' and B by A' throughout, we have:

If $B' \subseteq A'$, then $(A')' \subseteq (B')'$. Since $(A')' = A$ and $(B')' = B$, it follows that:

If $B' \subseteq A'$, then $A \subseteq B$.

The proofs of theorems in *c* and *e* are left as exercises.

5.6 SIMPLIFYING, FACTORING, AND MULTIPLYING POLYNOMIALS

The term "set polynomial" is employed to mean an expression created from sets or their complements by use of a finite number of operations chosen from union and intersection, as exemplified by the set polynomials A, $A \cap B$, $A' \cup B$, $(A' \cup B) \cap C$, $A \cup B \cap C \cap D$, $(A \cup B)' \cup C$. A set polynomial consists of parts called terms, each term being separated from the other by the operation of union. To be more specific, a term is any expression which consists of either a single letter representing a set or two or more such letters combined by the operation of intersection. (Auxiliary symbols such as parentheses may be used for grouping purposes.) Thus A, A', $A \cap B$, $A' \cap B \cap C$, and $(A \cup B) \cap (C \cap D)$ represent single terms. The polynomials $A' \cup B$ and $A \cup (B \cap C)$ consist of two terms, while $(A' \cup B) \cup C \cup D$ consists of three terms.

If we examine a set polynomial and consider intersection analogous with multiplication, union analogous with addition, and sets $A, B, C, \ldots$ analogous with real numbers $x, y, z, \ldots$, then the method for judging the number of terms in a polynomial involving sets is the same as that used for polynomials in the algebra of real numbers. If a set polynomial involves the intersection of sets, then each such set of the intersection is called a factor. The polynomial $(B \cup C' \cup D) \cap A' \cap (B \cup C)$ contains three factors, namely, $B \cup C' \cup D$, A', and $B \cup C$. It should be noted that a one-termed polynomial such as A may be regarded as $A \cap U$ with A and U as its factors.

The notion of a set polynomial is extended to include the cases where $n \geq 2$, $n \in \{\text{natural numbers}\}$, and $A_1, A_2, \ldots, A_n$ represent n sets with elements $\in U$. Here $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$ is used to mean the set of elements common to all the sets A_1 to A_n inclusive, where $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$ is regarded as a one-termed polynomial. Further, $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ represents the totality of all the elements of the sets A_1 to A_n inclusive, where $A_1 \cup A_2 \cup \cdots \cup A_n$ is regarded as an n-termed polynomial. De Morgan's laws can be extended and are stated here without proof:

$$(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)' = (A_1)' \cap (A_2)' \cap (A_3)' \cap \cdots \cap (A_n)'$$
$$(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n)' = (A_1)' \cup (A_2)' \cup (A_3)' \cup \cdots \cup (A_n)'$$

Correspondingly, the distributive laws take on the general forms

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$
$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$$

The following examples illustrate the use of the laws of the algebra of sets for operating with polynomials.

Example 1. Expand $(A' \cup C) \cap (B' \cup D')$.

	Authority
$(A' \cup C) \cap (B' \cup D')$	
$= [(A' \cup C) \cap B'] \cup [(A' \cup C) \cap D']$	Law $8b$
$= [B' \cap (A' \cup C)] \cup [D' \cap (A' \cup C)]$	Law $6b$
$= (B' \cap A') \cup (B' \cap C) \cup (D' \cap A') \cup (D' \cap C)$	Law $8b$

Note that the original expression contains one term while the final expression contains four terms.

Example 2. Factor each of the following:

a. $(A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$
b. $A \cup [B \cap (C \cup A')]$
c. $(A \cap B) \cup (C \cap D)$

	Authority
a. $(A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$	
$= [A \cap (C \cup D)] \cup [B \cap (C \cup D)]$	Law $8b$
$= [(C \cup D) \cap A] \cup [(C \cup D) \cap B]$	Law $6b$
$= (C \cup D) \cap (A \cup B)$	Law $8b$
b. $A \cup [B \cap (C \cup D)]$	
$= (A \cup B) \cap (A \cup C \cup D)$	Law $8a$
c. $(A \cap B) \cup (C \cap D)$	
$= [(A \cap B) \cup C] \cap [(A \cap B) \cup D]$	Law $8a$
$= [C \cup (A \cap B)] \cap [D \cup (A \cap B)]$	Law $6a$
$= (C \cup A) \cap (C \cup B) \cap (D \cup A) \cap (D \cup B)$	Law $8a$

Example 3. Simplify each of the following expressions:

a. $(A \cup B)' \cup (A' \cap B)$
b. $B \cup [(A' \cup B) \cap A]'$
c. $A \cup (A \cap B)$

	Authority
a. $(A \cup B)' \cup (A' \cap B)$	
$= (A' \cap B') \cup (A' \cap B)$	Law $9a$
$= A' \cap (B' \cup B)$	Law $8b$
$= A' \cap (U)$	Law $4a$
$= A'$	Law $2b$
b. $B \cup [(A' \cup B) \cap A]'$	
$= B \cup [(A' \cup B)' \cup A']$	Law $9b$
$= B \cup \{[(A')' \cap B'] \cup A\}$	Law $9a$
$= B \cup [(A \cap B') \cup A']$	Law $5a$
$= B \cup [A' \cup (A \cap B')]$	Law $6a$
$= B \cup [(A' \cup A) \cap (A' \cup B')]$	Law $8a$
$= B \cup [U \cap (A' \cup B')]$	Law $4a$
$= B \cup (A' \cup B')$	Laws $6b$ and $2b$
$= B \cup (B' \cup A')$	Law $6a$
$= (B \cup B') \cup A'$	Law $7a$
$= U \cup A'$	Law $4a$
$= U$	Laws $6b$ and $2a$

c. $A \cup (A \cap B)$

$= (A \cap U) \cup (A \cap B)$	Law 2*b*
$= A \cap (U \cup B)$	Law 8*b*
$= A \cap U$	Laws 6*a* and 2*a*
$= A$	Law 2*b*

Frequently, these simplification procedures provide a means for modifying polynomials that arise in other mathematical systems or models which have the same structure as the algebra of sets. These simplified versions of polynomials are often more manageable in the light of their interpretation to a specific mathematical model.

5.7 THE DUALITY PRINCIPLE

In the various laws for an algebra of sets it should be observed that if $\emptyset$ is replaced by U, U by $\emptyset$, $\cup$ by $\cap$, and $\cap$ by $\cup$ wherever these occur, then the resulting statement is again a law of the algebra of sets. This property is referred to as the "duality principle," and any new law formed as a result of its application is the "dual" of the original law. For example, the dual of a statement such as $A = (A \cap B) \cup (A \cap B')$ becomes $A = (A \cup B) \cap (A \cup B')$. The validity of this dual may be verified by the methods illustrated in Sections 5.4 and 5.5. The original statement $A = (A \cap B) \cup (A \cap B')$ was proved in Section 5.5.

Exercise 21

1. Verify each of the laws in Section 5.3 by means of Venn diagrams and/or membership tables.

2. If A, B, and C are subsets of U, prove the following identities by use of membership tables:

a. $(A' \cup B')' = A \cap B$
b. $A \cap (A \cap B)' = A \cap B'$
c. $(A' \cap B) \cap (A \cap B') = \emptyset$
d. $(A \cap B') \cap B' = A \cap B'$
e. $[A' \cup (A \cap B)]' = A \cap B'$
f. $(A' \cup B')' \cup (A' \cup B)' = A$
g. $[A' \cup (B \cup C)]' = A \cap B' \cap C'$
h. $A \cup (A' \cap B) = A \cup B$
i. $(A \cup B') \cap (A' \cup B) = (A' \cap B') \cup (B \cap A)$
j. $(A' \cup B)' = (A \cup B') \cap (A \cup B) \cap (A' \cup B')$

3. Using the laws for the "algebra of sets," perform the following operations:

a. Expand:
(1) $A \cup (B' \cap C')$
(2) $A' \cap (B' \cup C)$
(3) $(A' \cup B) \cap (C \cup D)$
(4) $(A' \cup B') \cap (C' \cup D' \cup E')$

b. Factor:
(1) $(A \cap B) \cup (A \cap C)$
(2) $A \cup (B \cap C')$
(3) $A' \cup B \cap (C \cup D)$
(4) $(A \cap B) \cup (C \cap D)$
(5) $[(A \cap B) \cap (C \cup D)] \cup [A \cap C']$

c. Simplify:
(1) $A \cup A' \cup B$
(2) $A \cap A' \cap C$
(3) $A \cap B \cap A' \cap B'$
(4) $(A \cup B \cup A' \cup B')'$

(5) $[(A \cap B) \cup (A \cap B') \cup (A' \cap B) \cup (A' \cap B')]'$
(6) $(A \cup B') \cap (A \cup B \cup C) \cap (A \cup C)$
(7) $(A \cap B)' \cap (A' \cup B)$ (8) $B \cap [(A' \cap B) \cup A]'$
(9) $A \cap (A \cup B)$
(10) $[(A \cup B) \cap (A' \cup C)] \cup [B \cap (B \cup C')]$

4. By use of the laws of the algebra of sets, prove each of the identities in Problem 2. Justify each step in your proof.

5. Write the dual of each of the statements in Problem 2.

6. If A and B are subsets of a universal set U, then:

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$
$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$
$$A' = \{x \in U \mid x \notin A\}$$

Using the symbols "$\in$", "$\notin$," "$\vee$," and "$\wedge$," rewrite each of the following sets:
a. $A \cap B'$ *b.* $A' \cup B'$ *c.* $(A' \cap B') \cup C$
d. $A' \cap B'$

Example. $(A \cap B) \cup C' = \{x \in U \mid (x \in A \wedge x \in B) \vee x \notin C\}$

7. Draw Venn diagrams to illustrate each of the following:
a. $A \subseteq (A \cup B)$ *b.* $(A \cap B) \subseteq A$
c. $A \subseteq B$ implies $A \cap B = A$ *d.* $(A \cap B') \subseteq A$
e. $A \subseteq B$ implies $A \cup B = B$ *f.* $A \subseteq B'$ implies $A \cap B = \emptyset$
g. $A' \subseteq B$ implies $A \cup B = U$

5.8 BOOLEAN ALGEBRA

The algebra of sets can be considered a model of the abstract mathematical system called a "Boolean algebra." The elements, relations, and operations of the algebra of sets obey the same laws as those of a Boolean algebra. In fact, every abstract Boolean algebra with either a finite or an infinite number of elements has an algebra of sets as a model. As a means of introducing the notions of a Boolean algebra, a finite algebra of sets containing four elements is illustrated.

Example 1. If $U = \{a,b,c\}$ and if $A = \{a,b\}$, then $A' = \{c\}$. The set of subsets $T = \{U,A,A',\emptyset\}$, with $\cap$, $\cup$, and $'$, forms an algebra of sets (a model of a Boolean algebra). The following three tables list all the possible unions and intersections of these four sets and their complements.

$\cup$	$\emptyset$	A	A'	U
$\emptyset$	$\emptyset$	A	A'	U
A	A	A	U	U
A'	A'	U	A'	U
U	U	U	U	U

$\cap$	U	A	A'	$\emptyset$
U	U	A	A'	$\emptyset$
A	A	A	$\emptyset$	$\emptyset$
A'	A'	$\emptyset$	A'	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

	$'$
A	A'
$\emptyset$	U
A'	A
U	$\emptyset$

The following illustrate the construction of the tables:

$$\begin{aligned} A \cup A' &= \{a,b\} \cup \{c\} \\ &= \{a,b,c\} \\ &= U \end{aligned} \qquad \begin{aligned} A' \cap U &= \{c\} \cap \{a,b,c\} \\ &= \{c\} \\ &= A' \end{aligned}$$

An examination of the tables reveals that:

1. The property of closure holds for $\cup$, $\cap$, and $'$.
2. The binary operations of $\cup$ and $\cap$ are commutative and associative.
3. The operation $\cup$ is distributive over $\cap$, and $\cap$ is distributive over $\cup$.
4. The identity elements for $\cup$ and $\cap$ are $\emptyset$ and U, respectively.
5. The complement laws hold, since

$$\begin{aligned} A \cup A' &= U \\ A' \cup A &= U \\ \emptyset \cup U &= U \\ U \cup \emptyset &= U \end{aligned} \qquad \begin{aligned} A \cap A' &= \emptyset \\ A' \cap A &= \emptyset \\ \emptyset \cap U &= \emptyset \\ U \cap \emptyset &= \emptyset \end{aligned}$$

Note that the complement of U is $\emptyset$ and vice versa.

6. As a means of illustrating some of the other laws of the algebra of sets, the following examples are included.

a. The testing of the distributive law for the particular choice of elements A, A', and $\emptyset$ shows that $A \cap (A' \cup \emptyset) = (A \cap A') \cup (A \cap \emptyset)$.

To prove this statement, the tables are used and each member is treated independently.

$$\begin{aligned} A \cap (A' \cup \emptyset) &= A \cap A' \\ &= \emptyset \end{aligned} \qquad \text{and} \qquad \begin{aligned} (A \cap A') \cup (A \cap \emptyset) &= \emptyset \cup \emptyset \\ &= \emptyset \end{aligned}$$

b. The testing of De Morgan's law for the particular choice of elements A and A' shows that $(A \cup A')' = A' \cap (A')'$. Here

$$\begin{aligned} (A \cup A')' &= (U)' \\ &= \emptyset \end{aligned} \qquad \text{and} \qquad \begin{aligned} A' \cap (A')' &= A' \cap A \\ &= \emptyset \end{aligned}$$

In summary, this model (algebra of sets) is designated as $\{T: \cup,\cap,'\}$, where $T = \{U,A,A',\emptyset\}$ and $\cup$, $\cap$, and $'$ refer to union, intersection, and complementation.

The elements of many models of Boolean algebras are not necessarily interpreted as being sets. To describe more generally the mathematical system called a Boolean algebra, the algebra of sets may be used as a guide with A, B, C, D, . . . , now representing abstract elements, $\emptyset$ and U now replaced by 0 and 1, the operations $\cup$ and $\cap$ now replaced by $+$ and $\cdot$, respectively, and $'$ left unchanged. It should be kept in mind that the operations of $+$ and $\cdot$ are defined for the particular model under consideration and, even though called addition and multiplication,

should not be confused with the meanings usually associated with these symbols. Further, the elements 0 and 1 have only those properties prescribed by the laws.

If these designated replacements are made in Laws 1 to 9 for the algebra of sets in Section 5.3, then the laws become those for a Boolean algebra.

Identity Laws

1a. $A + 0 = A$ 1b. $A \cdot 0 = 0$
2a. $A + 1 = 1$ 2b. $A \cdot 1 = A$

Idempotent Laws

3a. $A + A = A$ 3b. $A \cdot A = A$

Complement Laws

4a. $A + A' = 1$ 4b. $A \cdot A' = 0$
5a. $(A')' = A$ 5b. $0' = 1;\ 1' = 0$

Commutative Laws

6a. $A + B = B + A$ 6b. $A \cdot B = B \cdot A$

Associative Laws

7a. $(A + B) + C = A + (B + C)$
7b. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

Distributive Laws

8a. $A + (B \cdot C) = (A + B) \cdot (A + C)$
8b. $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$

De Morgan's Laws

9a. $(A + B)' = A' \cdot B'$ 9b. $(A \cdot B)' = A' + B'$

It is noted that the principle of duality, discussed in Section 5.7, still retains its significance throughout this new list of laws.

Thus, if a Boolean algebra of four elements $B = \{0,A,A',1\}$, where A and A' are abstract elements (not necessarily sets), is being considered, then the following tables define $+$, $\cdot$, and $'$.

$+$	0	A	A'	1
0	0	A	A'	1
A	A	A	1	1
A'	A'	1	A'	1
1	1	1	1	1

$\cdot$	1	A	A'	0
1	1	A	A'	0
A	A	A	0	0
A'	A'	0	A'	0
0	0	0	0	0

	$'$
A	A'
0	1
A'	A
1	0

This system is designated as $\{B: +,\cdot,'\}$.

Example 2. An interesting example of a model of a Boolean algebra of four elements with $+$, $\cdot$, and $'$ assigned different meanings is illustrated by the set $D = \{1,2,5,10\}$, where the elements of D represent the positive integral divisors of 10. For any $a,b \in D$, $a + b$ will be interpreted to mean the least common multiple (LCM) of a and b, $a \cdot b$ to mean the greatest common divisor (GCD) of a and b, and a' to mean the quotient when 10 is divided by a. Hence

$+$	1	2	5	10
1	1	2	5	10
2	2	2	10	10
5	5	10	5	10
10	10	10	10	10

$\cdot$	10	5	2	1
10	10	5	2	1
5	5	5	1	1
2	2	1	2	1
1	1	1	1	1

a	a'
2	5
1	10
5	2
10	1

For example, $2 + 10 = 10$, $5 + 5 = 5$, $1 + 1 = 1$, $10 \cdot 2 = 2$, $5 \cdot 2 = 1$, and $1' = 10$. This system is designated as $\{D: +,\cdot,'\}$.

To illustrate the distributive law $A + (B \cdot C) = (A + B) \cdot (A + C)$ according to the present interpretation of $+$ and $\cdot$, then:

$$\begin{aligned} 2 + (5 \cdot 10) &= 2 + 5 \\ &= 10 \end{aligned} \qquad \text{and} \qquad \begin{aligned} (2 + 5) \cdot (2 + 10) &= 10 \cdot 10 \\ &= 10 \end{aligned}$$

If the tables for the set $B = \{1,A,A',0\}$ and the set $T = \{U,A,A',\emptyset\}$ are compared with those for the set $D = \{1,2,5,10\}$, a similarity of structure is observed. The existing 1-1 correspondences are shown in Table 1.

Table 1

Elements			Operations		
B	T	D	B	T	D
$1 \leftrightarrow$	$U \leftrightarrow$	10	$+ \leftrightarrow$	$\cup \leftrightarrow$	$+$
$A \leftrightarrow$	$A \leftrightarrow$	2	$\cdot \leftrightarrow$	$\cap \leftrightarrow$	$\cdot$
$A' \leftrightarrow$	$A' \leftrightarrow$	5	$' \leftrightarrow$	$' \leftrightarrow$	$'$
$0 \leftrightarrow$	$\emptyset \leftrightarrow$	1			

If in the tables for $\{T: \cup,\cap,'\}$, U is replaced by 10, A by 2, A' by 5, and $\emptyset$ by 1, and the operations $\cup$ and $\cap$ are replaced by $+$ and $\cdot$, the tables for $\{D: +,\cdot,'\}$ result. As a consequence, any statement involving the elements of T will have a corresponding statement involving the elements of D. For example,

$$\begin{aligned} (A \cup A') \cap A &= U \cap A \\ &= A \text{ in } \{T: \cup,\cap,'\} \end{aligned} \qquad \text{and} \qquad \begin{aligned} (2 + 5) \cdot 2 &= 10 \cdot 2 \\ &= 2 \text{ in } \{D: +,\cdot,'\} \end{aligned}$$

These statements may be checked by using the respective tables for D and T.

The two interpretations $\{D: +,\cdot,'\}$ and $\{T: \cup,\cap,'\}$ of the Boolean algebra $\{B: +,\cdot,'\}$ are said to be isomorphic. This follows since the correspondence existing between the elements of the two models holds under all operations.

5.9 THE BINARY BOOLEAN ALGEBRA OF THE TWO ELEMENTS 0 AND 1

The simplest Boolean algebra contains only the two elements 0 and 1; i.e., the variables $A, B, C, D, \ldots$ in the list of laws of Section 5.8 take on only the values 0 and 1. This Boolean algebra is referred to as a binary Boolean algebra with $+$, $\cdot$, and $'$ defined on any A and B according to the following tables:

A	B	$A + B$	$A \cdot B$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

A	A'
1	0
0	1

The algebra of sets consisting of the two elements U and $\emptyset$ is a model of a binary Boolean algebra. Its tables for union, intersection, and complementation are:

A	B	$A \cup B$	$A \cap B$
U	U	U	U
U	$\emptyset$	U	$\emptyset$
$\emptyset$	U	U	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

A	A'
U	$\emptyset$
$\emptyset$	U

The abstract mathematical system referred to as the binary Boolean algebra satisfies all the laws of Section 5.8. For example, De Morgan's law (9a) is verified as shown in Table 1. The columns enclosed by

Table 1

A	B	A'	B'	$A + B$	$(A + B)'$	$A' \cdot B'$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

double lines represent the left and right members of De Morgan's law, $(A + B)' = A' \cdot B'$.

Since the variables $A, B, C, D, \ldots$ are permitted to assume only the values 0 and 1, these tables exhibit all possibilities involving any two variables A and B with the two binary operations $+$ and $\cdot$. No difficulty should be encountered for accepting any of the statements defined by these tables, with the exception of $1 + 1 = 1$. However, one should recognize the similarity of this statement to "$U \cup U = U$," which, interpreted loosely, states that "everything in the universe in union with everything in the universe yields everything in the universe." It is also of interest to note that exponents or coefficients other than 0 and 1 are not needed in this type of algebra, since the result of the product $A \cdot A \cdot A \cdot \ldots \cdot A$ or the sum $A + A + A + \cdots + A$ will be either 0 or 1.

The tabular method of testing the validity of a law or theorem is analogous to the membership-table method used in previous sections. In a similar manner all the other laws may be tested by this method. Their verification is left as an exercise.

The methods of procedure for proving theorems in a binary Boolean algebra are illustrated by the inclusion of the following two proofs.

Example 1. Prove $A + AB = A$ (where $AB = A \cdot B$), as shown in Table 2. Columns 1 and 4 are identical, and the theorem is proved.

Table 2

A	B	AB	$A + AB$
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

Example 2. Prove $A + (A'C + B) = (A + B) + C$, as shown in Table 3.

Table 3

A	B	C	A'	$A'C$	$A'C + B$	$A + B$	$A + (A'C + B)$	$(A + B) + C$
1	1	1	0	0	1	1	1	1
1	1	0	0	0	1	1	1	1
1	0	1	0	0	0	1	1	1
1	0	0	0	0	0	1	1	1
0	1	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	0	0	0	0	0

The theorems of Examples 1 and 2 may also be proved by using the laws of a Boolean algebra. This is illustrated in Example 3.

Example 3. $A + (A'C + B) = (A + B) + C.$

Proof:	*Authority*
$A + (A'C + B) = (A + A'C) + B$	Law 7a
$= (A + A')(A + C) + B$	Law 8a
$= 1 \cdot (A + C) + B$	Law 4a
$= (A + C) + B$	Laws 6b and 2b
$= A + (C + B)$	Law 7a
$= A + (B + C)$	Law 6a
$= (A + B) + C$	Law 7a

5.10 THE "ALGEBRA OF CIRCUITS"

Two other models of a binary Boolean algebra which have received much attention recently are:

a. The "algebra of circuits" as an interpretation of a binary Boolean algebra that has application to the design and construction of electronic computers and dial telephone systems

b. The "algebra of propositions" as an interpretation of a binary Boolean algebra concerned with the study of "methods of reasoning"

Since both of these models are quite extensive and complex, the objective here is to introduce briefly the meanings of certain key terms and symbols pertinent to these interpretations.

The simplest type of circuit involves switches. Switches will be represented by single letters, such as A, B, C, ... A and A' will refer to two switches that operate simultaneously but have opposing states, namely, if A is open, then A' is closed and vice versa. If two switches operate so that both close or both open simultaneously, then the same letter will be used to designate each of these switches.

Two switches are said to be connected in parallel when current flows if either or both of the switches are closed. Two switches are said to be connected in series when current flows only if both are closed. If A and B represent two switches, then $A + B$ implies a parallel connection while AB implies a series connection. It is noted that $A + B$ has the same meaning as the disjunction "A or B," but AB has the same meaning as the conjunction "A and B."

Illustrations of these circuits are shown in Fig. 98, and they are summarized in Table 1. In the table an "open circuit" means that

current will not flow, while a "closed circuit" means that current will flow.

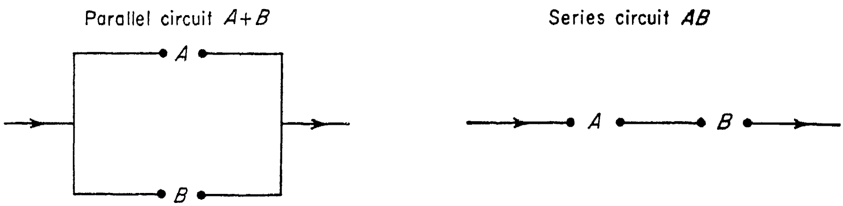


FIG. 98

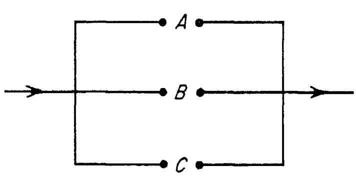
Table 1

Parallel			Series		
A	B	Circuit $A + B$	A	B	Circuit AB
Open	Open	Open	Open	Open	Open
Open	Closed	Closed	Open	Closed	Open
Closed	Open	Closed	Closed	Open	Open
Closed	Closed	Closed	Closed	Closed	Closed

If "open" is replaced by "0" and "closed" by "1," then the switches in parallel and the switches in series operate in accordance with the addition and multiplication tables, respectively, of the binary Boolean algebra. An isomorphism exists between the binary Boolean algebra and the algebra of electric circuits.

Example 1. Each of the switching networks shown in Figs. 99 and 100 has been translated into a Boolean polynomial, where a Boolean polynomial refers to any expression created from the symbols representing the variables and operations of a Boolean algebra. Since current will flow if one, two, or all of the switches are closed, then $A + B + C$ represents the network shown in Fig. 99.

Network *Boolean polynomial*



$A + B + C$

FIG. 99

Example 2. The networks of Fig. 100*b* and *c* are equivalent; that is, current will flow through each of the two circuits when the same combinations of switches A, B, and C are open or closed. Table 2 illustrates

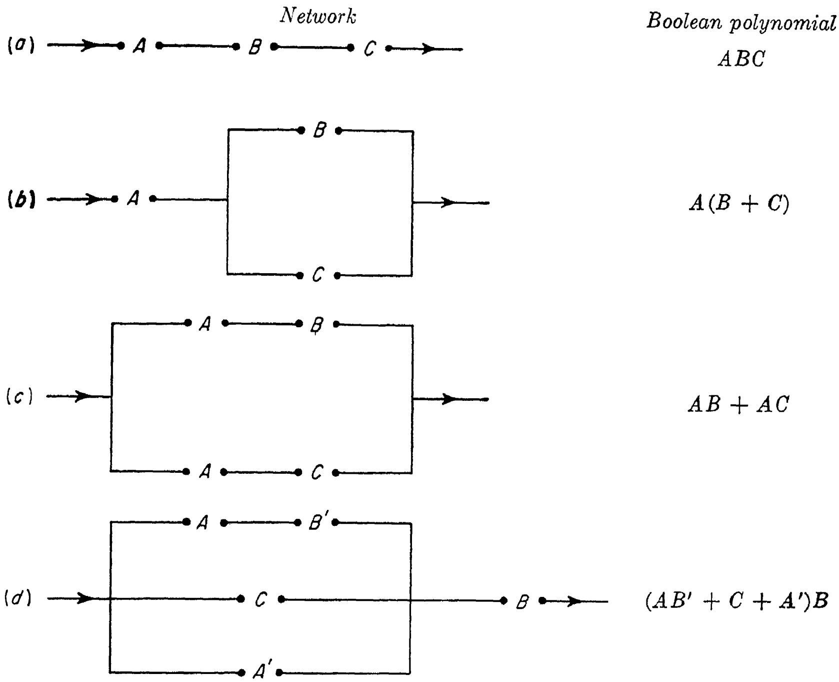


FIG. 100

Table 2

A	B	C	$A(B+C)$	$AB+AC$
Open	Open	Open	Open	Open
Open	Open	Closed	Open	Open
Open	Closed	Open	Open	Open
Open	Closed	Closed	Open	Open
Closed	Open	Open	Open	Open
Closed	Open	Closed	Closed	Closed
Closed	Closed	Open	Closed	Closed
Closed	Closed	Closed	Closed	Closed

this conclusion. Note that all eight possible combinations for A, B, and C have been examined.

The laws of a Boolean algebra frequently enable one to modify a given polynomial so that a simpler electric circuit may be designed. The

network $(AB' + C + A')B$ in Fig. 100d is equivalent to the network represented by the polynomial $CB + BA'$, which is a consequence of the laws of Boolean algebra as applied to $(AB' + C + A')B$. This follows since

		Authority
$(AB' + C + A')B$	$= B(AB' + C + A')$	Law 6b
	$= B(AB') + BC + BA'$	Law 8b
	$= B(B'A) + BC + BA'$	Law 6b
	$= (BB')A + BC + BA'$	Law 7b
	$= 0 \cdot A + BC + BA'$	Law 4b
	$= 0 + BC + BA'$	Laws 6b and 1b
	$= BC + BA'$	Laws 7a, 6b, and 1a

Further, $BC + BA' = B(C + A')$ and either of the circuits shown in Fig. 101 may be utilized to produce the same effect.

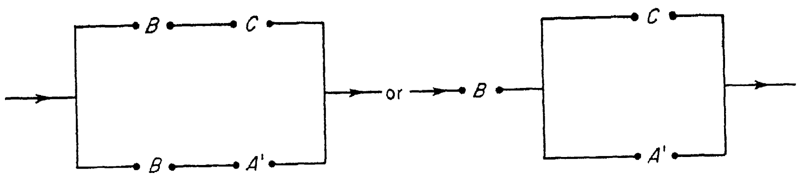


FIG. 101

Example 3. The network shown in Fig. 102 is controlled by a box with three buttons A, B, and C. If button A is moved to closed position, all the switches marked A will close and those marked A' will open. If button A is moved to open position, the switches marked A will open

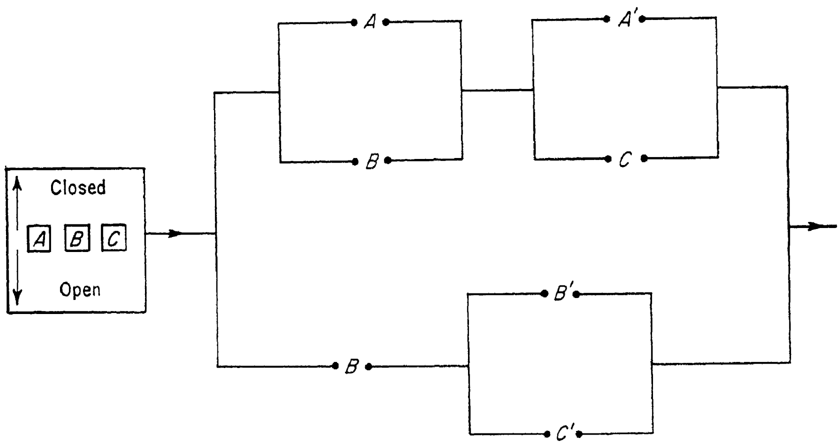


FIG. 102

and those marked A' will close. *Open position* for any button in the box closes its corresponding *primed switches*, and *closed position* for any button closes its corresponding *unprimed switches*.

A translation of the circuit shown in Fig. 102 in the language of Boolean algebra, where $+$ represents two switches in parallel and $\cdot$ represents two switches in series, is given by the polynomial $(A + B)(A' + C) + B(B' + C')$. This polynomial may be written in simpler form through the use of the laws of a Boolean algebra.

	Authority
$(A + B)(A' + C) + B(B' + C')$	
$= AA' + AC + A'B + BC + BB' + BC'$	Law 8b
$= 0 + AC + A'B + BC + 0 + BC'$	Law 4b
$= AC + A'B + BC + BC'$	Law 1a
$= AC + A'B + B(C + C')$	Law 8b
$= AC + A'B + B(1)$	Law 4a
$= AC + A'B + B$	Law 2b
$= AC + B$	Example 1, Section 5.9

The polynomial $AC + B$ results in the design of a new circuit which accomplishes the same purpose as the original circuit. Further,

$$AC + B = (A + B)(C + B)$$

and either of the circuits shown in Fig. 103 may be utilized to produce the same effect.

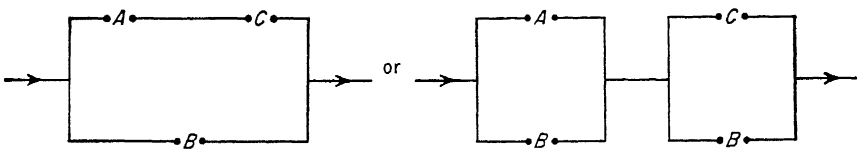


FIG. 103

Example 4. A light in a house is controlled by two switches. When both switches are off or when both are on the light is on; when one switch is on and the other is off, the light is off. Now if A and B represent the two switches in an "on" position and A' and B' represent the switches in an "off" position, the polynomial $A \cdot B + A' \cdot B'$ is equal to L, where L corresponds to the light being on. The plus sign is substituted for the word "or" and the multiplication sign for the word "and." Hence

	Authority
$AB + A'B' = (AB + A')(AB + B')$	Law 8a
$= (A + A')(B + A')(A + B')(B + B')$	Laws 6a and 8a
$= 1 \cdot (B + A')(A + B') \cdot 1$	Law 4a
$= (B + A')(A + B')$	Laws 2b and 6b

Either of the two circuits (Fig. 104) based on the two polynomials, $AB + A'B$ or $(B + A')(A + B')$, can be used to accomplish the desired result. Thus,

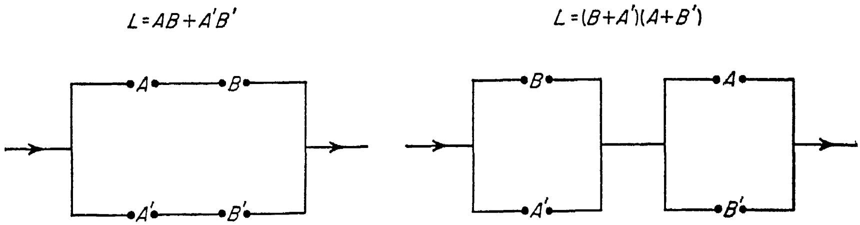


FIG. 104

It is suggested that the two equations

$$(A + B)(A' + C) + B(B' + C') = AC + B$$

and

$$AB + A'B' = (B + A')(A + B')$$

of Examples 3 and 4 be verified by the tabular procedure (see Section 5.9).

Example 5. Machines may be designed for the purpose of verifying decisions to be made when performing the moves of certain puzzles. It should be noted and stressed that a complete analysis of the proposed puzzle must be carried out before a machine can be designed. The completed machine does not render any decisions which the player himself cannot make, but the consequences of desired moves for the puzzle are usually obtained with greater speed. The following well-known and frequently repeated puzzle is now examined.

A man traveling with a goat, a wolf, and a basket of cabbages comes to a river that must be crossed. A boat is available that is large enough for the man and one of the other objects. The wolf and goat cannot be left together (on either side of the river) without the man, since the wolf will eat the goat; nor can the goat and the cabbages be left together without the man, since the goat will eat the cabbages. How can the man get across the river with his entire cargo intact? How can a machine be designed that will light a red light to indicate a dangerous situation or a green light to indicate a safe situation?

The machine is to have a box with four buttons; each button is to have two settings: "closed position" to represent that the associated

object is on the initial side of the river and "open position" to indicate that this object is on the other side of the river. The letters M, G, W, and C indicate that each object is on the initial side of the river, while the letters M', G', W', and C' indicate that these objects are on the opposite side of the river. If the situation $MGW'C'$ is analyzed, it follows that the man and goat are on the initial side and the wolf and cabbage on the opposite side. The circuit contains switches marked M, G, W, C, W', M', G', and C' controlled by the button settings. Accordingly, $MGW'C'$ means that buttons M and G are in closed position and buttons W and C are in open position. This results in closed switches for those marked M, G, W', and C' and in open switches for those marked M', G', W, and C.

A consideration of all possible situations results in 16 or 2^4 different cases, since four objects are involved with two possible locations for each object. A listing of these 16 cases is given in Table 3.

If D corresponds to the set of the six dangerous situations, then

$$D = MG'WC' + MG'W'C + MG'W'C' + M'GWC' + M'GW'C + M'GWC$$

If we simplify according to the laws of Boolean algebra,

$$\begin{aligned}
& & & \textit{Authority} \\
D &= MG'WC' + MG'W'C + MG'W'C' \\
&\qquad + M'GWC + M'GWC' + M'GW'C \\
&= M'W'GC + M'WGC' + M'WGC \\
&\qquad + MW'G'C' + MW'G'C + MWG'C' & & \text{Laws } 6a,\ 6b,\ 7a,\ 7b \\
&= M'G(W'C + WC' + WC) \\
&\qquad + MG'(W'C' + W'C + WC') & & \text{Law } 8b \\
&= M'G[W'C + W(C' + C)] \\
&\qquad + MG'[W'(C' + C) + WC'] & & \text{Law } 8b \\
&= M'G[W'C + W(1)] + MG'[W'(1) + WC'] & & \text{Law } 4a \\
&= M'G(W'C + W) + MG'(W' + WC') & & \text{Law } 2b \\
&= M'G[(W' + W)(W + C)] \\
&\qquad + MG'[(W + W')(W' + C')] & & \text{Laws } 6a \text{ and } 8a \\
&= M'G(W + C) + MG'(W' + C') & & \text{Laws } 4a \text{ and } 2b
\end{aligned}$$

If we examine D', the set of all safe situations, and use De Morgan's law,

$$D' = [M'G(W + C) + MG'(W' + C')]'$$

Again simplifying according to the laws of Boolean algebra, we find

$$D' = [(M + G') + W'C'][(M' + G) + WC]$$

Table 3

Man	Goat	Wolf	Cabbage	Dangerous	Safe
M	G	W	C		x
M	G	W	C'		x
M	G	W'	C		x
M	G	W'	C'		x
M	G'	W	C		x
M	G'	W	C'	x	
M	G'	W'	C	x	
M	G'	W'	C'	x	
M'	G	W	C	x	
M'	G	W	C'	x	
M'	G	W'	C	x	
M'	G	W'	C'		x
M'	G'	W	C		x
M'	G'	W	C'		x
M'	G'	W'	C		x
M'	G'	W'	C'		x

The complete circuit for D and D' is shown in Fig. 105. The circuit shown in Fig. 106 illustrates a dangerous case, and that in Fig. 107 illustrates a safe case.

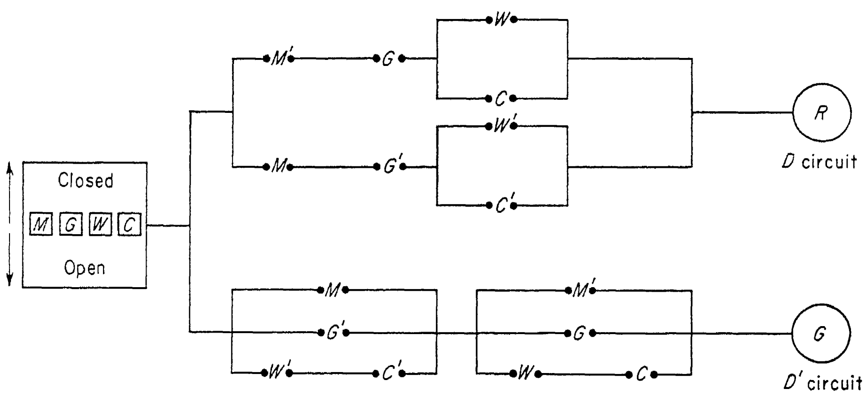


FIG. 105

5.11 THE "ALGEBRA OF PROPOSITIONS"

As indicated in the previous section, another model that has wide application is the "algebra of propositions." A proposition (simple statement) is a declarative statement; e.g., "$2 + 2 = 4$" or "It is

A dangerous case: $MG'WC'$

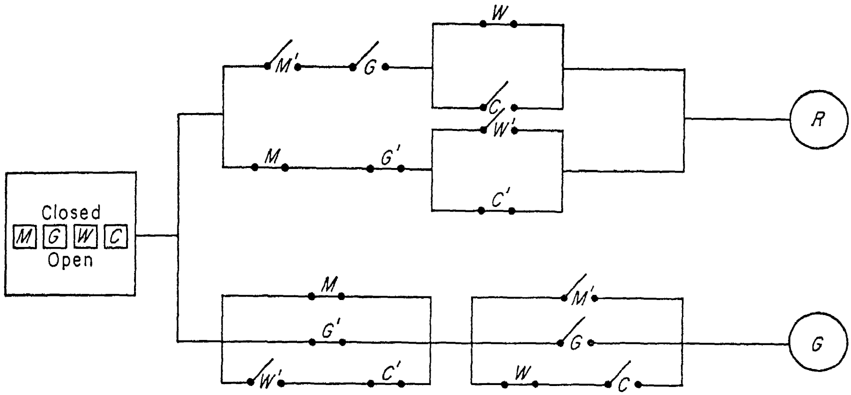


FIG. 106

A safe case: $M'G'W'C'$

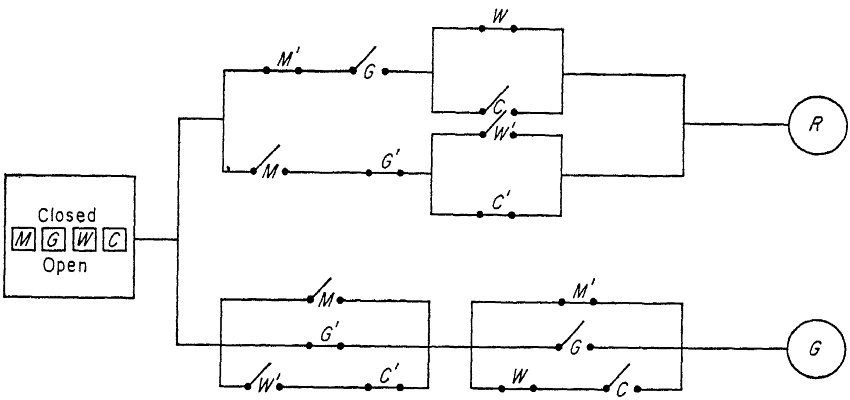


FIG. 107

raining." Only those statements that have a definite truth value will be considered in the algebra of propositions; i.e., each simple statement may be labeled as being definitely true or definitely false. Statements such as "There are living creatures on Venus" will not be elements of the chosen universe of statements because of their doubtful truth value. Hence all statements will assume one of two values, T (true) or F (false). The elements of the algebra will be simple statements designated by the small letters p, q, r, ... The operations $\wedge$, $\vee$, and $'$, meaning "and," "or," and "not," respectively, are used with simple sentences to form compound sentences and are analogous to $\cdot$, $+$, and $'$ of the binary Boolean algebra.

The compound statement $p \wedge q$ is called the "conjunction" of p

and q; it is true only when *both* p and q are true and is false otherwise. The compound statement $p \vee q$ is called the "disjunction" of p and q; it is true when at least one of the simple statements is true and is false only when both p and q are false. The negation of p, written p', is false when p is a true statement and true when p is a false statement. These ideas are summarized in the following "truth tables."

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	p'
T	F
F	T

Example 1. If p and q represent the simple statements

p: John loves Mary.
q: It is raining.

then

$p \wedge q$: John loves Mary and it is raining.
$p \vee q$: John loves Mary or it is raining.
p': John does not love Mary.

Example 2. If r and s represent the simple statements

r: a is an integer ($a \in I$).
s: a is a prime.

then

$r \wedge s$: $a \in I$ and a is a prime.
$r \vee s$: $a \in I$ or a is a prime.
s': a is not a prime.
r': $a \notin I$.

Suppose that a is 5. Then $r \wedge s$ is a true statement, since both r and s are true. It follows that $r \vee s$ is also true and r' is false. If a is 4, then $r \wedge s$ is a false statement, $r \vee s$ is true, and s' is true.

Another connective that is utilized to form compound statements of the "If, then" type is the implication or conditional, denoted by $\rightarrow$. The "If, then" type is one of the most frequently used statements in mathematics. The truth table for implication is:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If p is the hypothesis and q the conclusion, then the implication $p \rightarrow q$ is false only when a false conclusion has been obtained from a true hypothesis. Example 3 serves to illustrate the truth table for implication.

Example 3. Suppose an instructor makes the following statement to a student: "If you receive a passing grade on the final examination, then you will pass the course." The student agrees to abide by this statement. Using the following translations, consider all four possible cases with respect to the truth table.

p: You receive a passing grade on the final examination.
q: You will receive credit for the course.
$p \rightarrow q$: If you receive a passing grade on the final examination, then you will receive credit for the course.

Case 1: Suppose p is true and q is true; i.e., you do receive a passing grade and you do receive credit. You will certainly have no cause to be angry with your instructor; i.e., $p \rightarrow q$ will be accepted as being true.

Case 2: Suppose p is true and q is false; i.e., you do receive a passing grade but still fail the course. You will certainly have cause to be angry. Here $p \rightarrow q$ will be accepted as being false.

Case 3: Suppose p is false and q is true; i.e., you do fail the examination but still pass the course. You certainly have no cause to be angry and $p \rightarrow q$ will be accepted as being true.

Case 4: Suppose p is false and q is false; i.e., you do not pass the final examination and do not receive credit in the course. You have no cause to be angry, since this was the original agreement. Hence $p \rightarrow q$ will be accepted as being true.

In plane geometry the terms "theorem," "converse," "inverse," and "contrapositive" were introduced. These terms were used in the construction of mathematical proofs of statements in the form "If p, then q," where p and q represented the hypothesis and conclusion, respectively, of the original theorem. Through a sequence of logical steps permitted by a stipulated mathematical system, a valid conclusion was usually obtained from some stated hypothesis. Theorems such as "If the diagonals of a quadrilateral are equal, then the quadrilateral is a square" or "If two parallel lines are cut by a transversal, then alternate interior angles are equal" were proved within the framework of the elements, postulates, and other theorems of plane geometry. In many instances the converse or inverse or contrapositive of the original theorem was employed to arrive at a valid conclusion. The meanings of these terms as related to an initially stated theorem are summarized in Table 1.

Table 1

	Statement	Symbolic representation
Theorem	If p, then q	$p \to q$
Converse	If q, then p	$q \to p$
Inverse	If not p, then not q	$p' \to q'$
Contrapositive	If not q, then not p	$q' \to p'$

Example 4. These ideas are illustrated with respect to the Venn diagram shown in Fig. 108. It is evident in this diagram that $A \subset B$.

Theorem:	If $a \in A$, then $a \in B$.	True
Converse:	If $a \in B$, then $a \in A$.	False
Inverse:	If $a \notin A$, then $a \notin B$.	False
Contrapositive:	If $a \notin B$, then $a \notin A$.	True

Notice that the initially stated theorem and its contrapositive are both true, while the inverse and the converse of the theorem are both false.

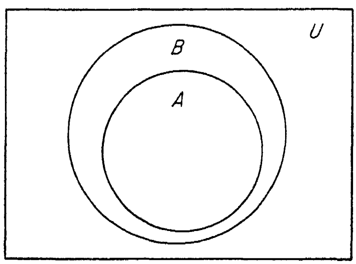


FIG. 108

Suppose the initial theorem was

If $a \in B$, then $a \in A$. False

Then

Converse:	If $a \in A$, then $a \in B$.	True
Inverse:	If $a \notin B$, then $a \notin A$.	True
Contrapositive:	If $a \notin A$, then $a \notin B$.	False

These examples imply that a theorem and its contrapositive are equivalent, while the inverse and converse of this same theorem are also equivalent; i.e., the theorem implies the contrapositive and vice versa, while the converse implies the inverse and vice versa. Symbolically

these facts are represented as follows:

Theorem	*Contrapositive*
$(p \to q) \leftrightarrow$	$(q' \to p')$
Converse	*Inverse*
$(q \to p) \leftrightarrow$	$(p' \to q')$

The symbol $\leftrightarrow$ is called a double-implication or equivalence symbol. The previous examples have illustrated that a compound statement involving the equivalence symbol is true when the left and right members both have the same truth value, i.e., when both are true or both are false. The truth table for equivalence is:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that $p \leftrightarrow q$ is read "If p then q, and if q then p" or "p if and only if q."

The following examples serve to illustrate the use of truth tables in examination of compound statements or compound expressions.

Example 5. The statement $(p \to q) \leftrightarrow (q' \to p')$ is verified in the manner shown in Table 2. The table dictates that the statement $p \to q$ has

Table 2

p	q	q'	p'	$p \to q$	$q' \to p'$	$(p \to q) \leftrightarrow (q' \to p')$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

the same truth value as the statement $q' \to p'$. As a consequence, these two statements are equivalent and this equivalence produces T for each entry in the last column. Such a compound statement as $(p \to q) \leftrightarrow (q' \to p')$ is called a "tautology." A tautology is a statement that is true regardless of the truth or falsity of the initial statements p and q. In a similar manner, $(q \to p) \leftrightarrow (p' \to q')$ can also be shown to be a tautology. This verification is left as an exercise.

The importance of the tautology $(p \to q) \leftrightarrow (q' \to p')$ is that it forms one of the bases for the method of indirect proof. This suggests that if difficulty is experienced when proving $p \to q$, it might be simpler to

prove $q' \to p'$, the contrapositive of the initial theorem. Correspondingly, a similar situation holds true for studying the inverse as related to the converse.

The laws for a Boolean algebra (Section 5.8) may be restated by replacing $+$ with $\vee$, $\cdot$ with $\wedge$, and A, B, and C with p, q, and r. Further, 0 is defined to represent a false proposition, while 1 represents a true proposition. In any statement where the symbol "$=$" is replaced by "$\leftrightarrow$," it will mean that the left member of this statement has the same truth value as its right member regardless of what the truth values are for $p, q, r, s, \ldots$ appearing in it.

According to these agreed replacements, the identity law (1*a*), $A + 0 = A$, takes the form $p \vee 0 \leftrightarrow p$. Since 0 represents a proposition that is always false, then $p \vee 0$ is equivalent to p according to the meaning of disjunction. This can also be shown by the truth table:

p	0	$p \vee 0$	$p \vee 0 \leftrightarrow p$
T	F	T	T
F	F	F	T

Since each entry in the last column is T, the statement $p \vee 0 \leftrightarrow p$ is a tautology.

Laws 1*b*, 2*a*, 2*b*, 4*a*, 4*b*, and 5*b* are easily rewritten in the language of the "algebra of propositions," and this is left as an exercise. For example, Law 4*a* translated into symbolic language is $p \vee p' \leftrightarrow 1$, and the truth value of this statement is always true. This is substantiated by the truth table:

p	p'	1	$p \vee p'$	$p \vee p' \leftrightarrow 1$
T	F	T	T	T
F	T	T	T	T

The tautology $p \vee p'$ is referred to as the law of "the excluded middle" in logic, namely, that a proposition or its negation must be true.

The remaining laws of the Boolean algebra become:

3*a*. $p \vee p \leftrightarrow p$ 3*b*. $p \wedge p \leftrightarrow p$
5*a*. $(p')' \leftrightarrow p$
6*a*. $(p \vee q) \leftrightarrow (q \vee p)$ 6*b*. $(p \wedge q) \leftrightarrow (q \wedge p)$
7*a*. $[(p \vee q) \vee r] \leftrightarrow [p \vee (q \vee r)]$
7*b*. $[(p \wedge q) \wedge r] \leftrightarrow [p \wedge (q \wedge r)]$
8*a*. $[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$
8*b*. $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
9*a*. $(p \vee q)' \leftrightarrow (p' \wedge q')$
9*b*. $(p \wedge q)' \leftrightarrow (p' \vee q')$

All these newly formed statements are tautologies and may be verified by the use of truth tables. The verification of De Morgan's law (9b), $(p \wedge q)' \leftrightarrow (p' \vee q')$, is illustrated in Table 3. Since each entry

Table 3

p	q	$p \wedge q$	$(p \wedge q)'$	p'	q'	$p' \vee q'$	$(p \wedge q)' \leftrightarrow (p' \vee q')$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

in the last column is T, the statement $(p \wedge q)' \leftrightarrow (p' \vee q')$ is a tautology.

The mathematical systems referred to as the "algebra of circuits," the "algebra of propositions," and the "algebra of sets" (consisting only of the two elements U and $\emptyset$) are all models of the binary Boolean algebra. An isomorphism exists between all of these structures, since the correspondences shown in Table 4 prevail.

Table 4

Boolean algebra		Sets		Switches		Propositions
0	$\leftrightarrow$	$\emptyset$	$\leftrightarrow$	Open	$\leftrightarrow$	False
1	$\leftrightarrow$	U	$\leftrightarrow$	Closed	$\leftrightarrow$	True
$\cdot$	$\leftrightarrow$	Intersection	$\leftrightarrow$	Series	$\leftrightarrow$	Conjunction
$+$	$\leftrightarrow$	Union	$\leftrightarrow$	Parallel	$\leftrightarrow$	Disjunction

Exercise 22

1. Evaluate each of the following Boolean polynomials for those cases where A, B, and C are variables permitted to assume only the values 0 and 1.

Example:
a. $AB' + C$

A	B	C	B'	AB'	$AB' + C$
1	1	1	0	0	1
1	1	0	0	0	0
1	0	1	1	1	1
1	0	0	1	1	1
0	1	1	0	0	1
0	1	0	0	0	0
0	0	1	1	0	1
0	0	0	1	0	0

For example, in row 3, if A is 1, B is 0, and C is 1, then $AB' + C$ is 1.

b. $A + B'$ *c.* $(A + B)'$ *d.* $A + A'B'$
e. AB *f.* $(A + B) + AB$ *g.* $A + BC$
h. $(A + B)(A + C)$ *i.* $A(AB' + A) + AB$ *j.* $A[(B + C)' + A]$

2. Using the tabular method of Section 5.9, verify the following laws and theorems of a binary Boolean algebra:
a. $(A'B')' = A + B$ *b.* $(A'B)' = (AB')' + AB + A'B'$
c. $(A'B')' + (A'B)' = A$ *d.* $A(AB)' = AB'$
e. $(A + B)(A + B') = A$ *f.* $B[(C + D)' + B] = B$
g. $(A + B)(A' + C)(B + C) = (A + B)(A' + C)$

3. Show that the set $G = \{\triangle, \bigcirc, \square, \diamond\}$ with the operations + and ·, defined by the following tables, is a Boolean algebra.

+	□	△	○	◇
□	□	△	○	◇
△	△	△	△	△
○	○	△	○	△
◇	◇	△	△	◇

·	□	△	○	◇
□	□	□	□	□
△	□	△	○	◇
○	□	○	○	□
◇	□	◇	□	◇

4. If a and b are any elements of D, then $a + b$ will be defined to mean the least common multiple of a and b, $a \cdot b$ the greatest common divisor of a and b, and a' the quotient when N is divided by a. Determine whether the following constitute Boolean algebras with respect to the designated operations:
a. D = set of the integral divisors of $N = 6$
b. D = set of the integral divisors of $N = 12$
c. D = set of the integral divisors of $N = 30$

5. Show that the power set 2^I is an algebra of sets if $I = \{1,2,3\}$. Show that this algebra of sets is isomorphic to the Boolean algebra of the integral divisors of 30.

6. Using only the laws of a Boolean algebra, prove each of the identities in Problem 2.

7. Design an electrical network to represent each of the following polynomials without first performing any simplifications on the polynomials.
a. $(A + B)A'$ *b.* $AB + AC$
c. $(A + B)(A' + B)$ *d.* $AB + B$
e. $AB + A'B + AB'$ *f.* $(AB + A'B + A'B')C$
g. $ABC + A + BC$ *h.* $(A + B' + C)A' + (B + C')D$
i. $A[(B + C)D + C'(E + F)]$
j. $(B + AC')(B + A' + C) + C'D + B'D$

8. Write a polynomial for each of the circuits shown in Fig. 109.

9. Draw a pair of electrical networks to represent each of the laws and theorems in parts *a* to *e*.

Example. $(A + B)(A + C) = A + BC$ as shown in Fig. 110. In either of the networks shown in Fig. 110, current flows only when switch A is closed or when both B and C are closed. Hence the two circuits are equivalent.
a. $A + B = B + A$ *b.* $AB = BA$ *c.* $A(B + C) = AB + AC$
d. $A(A + B) = A$ *e.* $ABC + AB'C + AB'C + A'B'C = (A + B')C$

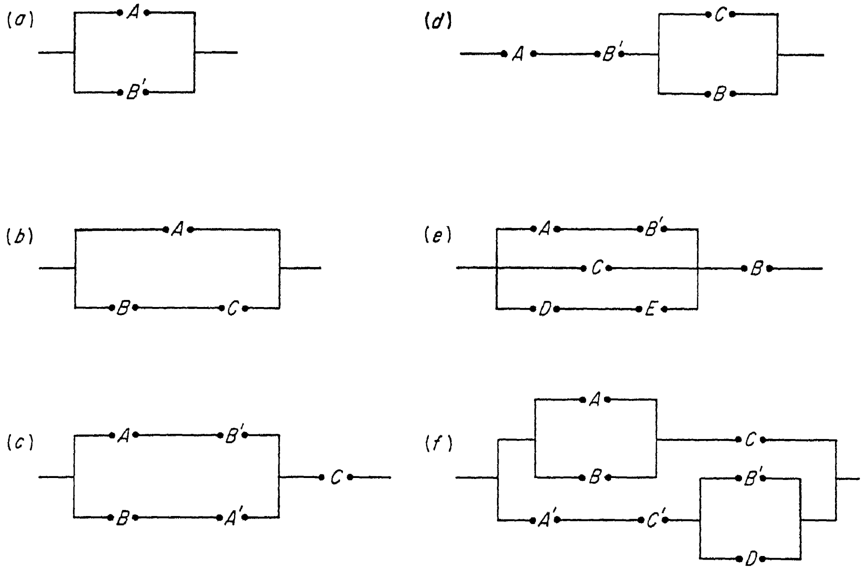


FIG. 109

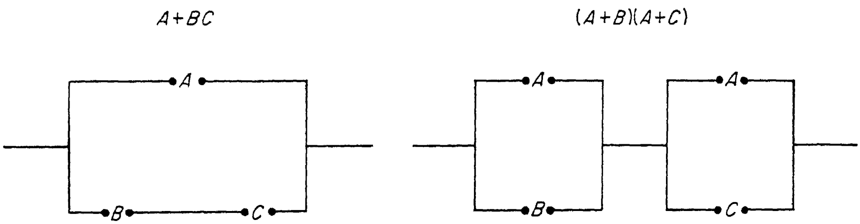


FIG. 110

10a. Construct a network for each of the following Boolean polynomials:

(1) BC (2) $[B(B + A)][C(B' + C)]$

b. By the use of the laws for a Boolean algebra, show that the networks of parts 1 and 2 are equivalent.

11. In each of the following compound statements, assign letters to the simple statements contained therein and then rewrite the original statement in symbolic form.

Example. Roses are red and violets are blue.

p:	Roses are red.
q:	Violets are blue.
$p \wedge q$:	Roses are red and violets are blue,

a. It is hot and it is humid.

b. Six is a whole number or $\frac{3}{4}$ is a fraction,

c, Jack and Jill went up the hill,

d. Roses are red or violets are blue.
e. Roses are not red or violets are not blue.
f. Roses are not red or violets are blue.
g. If I live in Ohio, then I live in the United States.
h. If a number is not even, then it is not divisible by 2.
i. A number is even if and only if it is divisible by 2.
j. A number is odd if and only if it is not divisible by 2.

12. Let p, q, and s be simple statements. If $p = 0$, $q = 1$, and $s = 0$, determine whether each of the compound statements in *a* to *j* is true or false ($p = 0$ means that p is false and $p = 1$ means that p is true).

a. $p \vee q$ *b.* $p' \vee q'$ *c.* $p' \wedge q$
d. $p \wedge p'$ *e.* $(p \wedge q) \vee s'$ *f.* $(p \wedge q) \vee (p' \wedge q')$
g. $(p \vee q) \wedge (p' \vee q')$ *h.* $p' \vee (q \vee s')'$
i. $(p \wedge q) \vee (p' \wedge q) \vee (p \wedge q') \vee (p' \wedge q')$
j. $[(p' \wedge q') \vee s] \wedge [(q' \vee p') \wedge s']$

13. Given: the simple statements

p: It is snowing.
q: The street is slippery.
r: My car has snow tires.
s: I arrive at work on time.

Write each of the statements in *a* to *j* in words.

a. $p \wedge q$ *b.* $q \wedge r$ *c.* $q \rightarrow p$
d. $p \rightarrow r$ *e.* $p' \rightarrow s$ *f.* $p' \vee q'$
g. $(p \wedge q) \rightarrow r$ *h.* $(p \wedge r) \rightarrow s$ *i.* $(p \wedge q \wedge r) \rightarrow s$
j. $s' \rightarrow r$

14*a*. Suppose that p and q are true and r and s are false. Obtain the truth value for each of the statements in Problem 13.

b. Suppose p and s are false and q and r are true. Obtain the truth value for each of the statements in Problem 13.

15. Verify the tautologies of Section 5.10 (3*a*, 3*b*, 5*a*, 6*a*, 6*b*, 7*a*, 7*b*, 8*a*, 8*b*, 9*a*, 9*b*) by the use of truth tables.

16. Which of the statements in *a* to *m* are tautologies?

a. $p' \rightarrow (p \rightarrow q)$ *b.* $(p \rightarrow q) \rightarrow (p' \rightarrow q')$
c. $(p')' \leftrightarrow p$ *d.* $[(p \vee q) \wedge r] \leftrightarrow [p \vee (q \wedge r)]$
e. $[(p \wedge q) \rightarrow r] \leftrightarrow [(p \wedge r') \rightarrow q']$ *f.* $[p \wedge (p \rightarrow q)] \rightarrow q$
g. $[(q' \rightarrow p') \wedge (q \rightarrow p)] \rightarrow (p \leftrightarrow q)$ *h.* $(p \rightarrow q)' \leftrightarrow (p \wedge q')$
i. $(p \rightarrow q) \leftrightarrow (p' \vee q)$ *j.* $[(p \wedge q) \vee r] \leftrightarrow [p \wedge (q \vee r)]$
k. $(p \wedge q) \rightarrow [p \leftrightarrow (q \vee s)]$ *l.* $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
m. $(p \leftrightarrow q) \leftrightarrow [(p \wedge q) \vee (p' \wedge q')]$

17. Write in words the converse, inverse, and contrapositive for each of the following theorems. Parts *a*, *d*, and *e* refer to plane geometry.

a. If a triangle is equilateral, then it is isosceles.
b. If it is snowing, then the street is slippery.
c. If a man lives in Boston, then he lives in Massachusetts.
d. If two angles are right angles, then they are equal.
e. If two lines do not intersect, then they are parallel.

18. If each implication in Problem 17 is assumed to be true, determine the truth or falsity of its converse, inverse, and contrapositive.

5.12 CONCEPT OF A GROUP

Among the abstract mathematical structures of importance are those of a group and a field. The basic characteristics of these structures are presented so as, first, to illustrate how other mathematical structures emanate from them and, second, to indicate that their postulates represent the framework from which well-known algebraic techniques arise. Certain aspects of the second objective have already been studied when they were discussed in Chapter 3 with reference to number systems. Other sets of numbers will now be examined under specified operations so as to determine whether they possess the properties of either a group, a field, or both. A group is defined as follows:

A group symbolized by $\{K: *\}$ is a mathematical structure which possesses a set of elements K, a well-defined binary operation $*$, and an equivalence relation between the elements of K. Further, it must satisfy the following properties:

G-1: The operation $*$ is closed.
For all $a \in K$ and $b \in K$, $a * b \in K$.

G-2: The operation $*$ is associative.
For all $a \in K$, $b \in K$, and $c \in K$, $(a * b) * c = a * (b * c)$.

G-3: There exists in K a unique element i, called the identity element, such that for all $a \in K$, $a * i = i * a = a$.

G-4: For every element $a \in K$, there exists a unique element $a' \in K$, called the inverse of a, such that $a * a' = a' * a = i$.

Though every element of K commutes with the identity element i, that is, $a * i = i * a$, it does not necessarily follow that $a * b = b * a$ for all $a,b \in K$. If the operation $*$ possesses the property of commutativity, then the group is referred to as a commutative or an abelian group. In most of the examples which follow, the groups will possess commutativity.

Example 1. The set of integers I with the operation $*$ interpreted as ordinary addition is a group. The identity element i is the integer zero, since for all $a \in I$, $a + 0 = 0 + a = a$. Further, for all $a \in I$, the inverse a' is $-a$, since $a + (-a) = (-a) + a = 0$. Note that 0 is its own inverse. The associative property represented by the statement $(a + b) + c = a + (b + c)$ is true, since this is valid for all integers a, b, and c. Therefore $\{I: +\}$, which is usually referred to as the "additive group of integers," is a group.

Example 2. The set of integers that are divisible by 2, namely

$$K = \{. . . , -8, -6, -4, -2, 0, 2, 4, 6, 8, . . .\}$$

with $*$ interpreted as ordinary addition, satisfies all the necessary properties of a group. This follows since the operation of addition is closed and associative with respect to K, and, further, 0 is the identity element. Each element in K has an inverse. For example, -2 is the inverse of 2, 4 is the inverse of -4, and 0 is its own inverse. Therefore $\{K: *\}$ is a group.

Example 3. The set of integers I under ordinary multiplication is not a group, since the multiplicative inverses of integers are not contained in I. That is, for every $a \in I$, except the element 1, there exists no inverse for a in I.

Example 4. The set of natural numbers, N, is not a group under the operation of addition, since N contains no inverse elements or identity element ($0 \notin N$).

Example 5. Table 1 exhibits additional sets of numbers where a specified operation has been considered with reference to the properties of a group. The structures in rows b and d are groups, while those in rows a, c, and e are not. If in the structures of rows c and e the zero element is excluded from consideration, these are also groups.

Table 1

	Set K	Operation $*$	Properties			
			G-1	G-2	G-3	G-4
$a.$	N	Multiplication	Yes	Yes	$i = 1$	No
$b.$	F	Addition	Yes	Yes	$i = 0$	Yes
$c.$	F	Multiplication	Yes	Yes	$i = 1$	0 has no inverse
$d.$	R_e	Addition	Yes	Yes	$i = 0$	Yes
$e.$	R_e	Multiplication	Yes	Yes	$i = 1$	0 has no inverse

Example 6. If a and r are integers and if $a - r$ is divided by a fixed integer $m > 1$ where the remainder is zero, then "a is congruent to r modulo m" and is written $a \equiv r(\text{mod } m)$. If m is 5, then $13 \equiv 3(\text{mod } 5)$, $25 \equiv 0(\text{mod } 5)$, and $-3 \equiv 2(\text{mod } 5)$, since $13 - 3$, $25 - 0$, and $-3\ -2$ are all divisible by 5. In general, if $a \equiv r(\text{mod } m)$ where $a,r \in I$ and m is an integer greater than 1, then $mq + r = a$, where q represents the quotient when a is divided by m, and r represents the remainder.

Suppose that every element of the set of integers is divided by 5 and

a listing is carried out according to the remainders 0, 1, 2, 3, and 4. A total of five subsets is obtained, namely:

$I_0 = \{. . . ,-10,-5,0,5,10, . . .\}$
$I_1 = \{. . . ,-9,-4,1,6,11, . . .\}$
$I_2 = \{. . . ,-8,-3,2,7,12, . . .\}$
$I_3 = \{. . . ,-7,-2,3,8,13, . . .\}$
$I_4 = \{. . . ,-6,-1,4,9,14, . . .\}$

These sets are subsets of I and are called equivalence classes. Note that

$$I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_4 = I$$

If $a \in I_2$, then $a \equiv 2(\text{mod } 5)$. For example, since $22 \in I_2$, then $22 \equiv 2(\text{mod } 5)$, because $22 - 2$ when divided by 5 leaves a remainder of zero. Similarly, since $-3 \in I_2$, then $-3 \equiv 2(\text{mod } 5)$ because the difference $-3 - 2 = -5$ when divided by 5 leaves a remainder of zero. Note that if 5 or any integral multiple of 5 is added to any element of an equivalence class, another distinct element of this same equivalence class is obtained. Further, it should be observed that the sum of an element of I_2 with an element of I_4 yields an element of I_1. Hence we write $I_2 \oplus I_4 = I_1$ with the understanding that the operation of addition defined in this way means that if any element of I_2 is combined with any element of I_4, an element of I_1 is obtained. For convenience and purposes of identification, $2 \oplus 4 = 1$ is used to replace $I_2 \oplus I_4 = I_1$. With these notions as a background, it is possible to develop an arithmetic referred to as a modular arithmetic.

Consider the set $K = \{0,1,2,3,4\}$ and the operation $*$ replaced by $\oplus$ and defined to be the remainder obtained when the sum of any two elements of K is divided by 5. For example, $4 \oplus 4 = 3$ and $3 \oplus 2 = 0$. The $\oplus$ table for the set K becomes

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

A study of this table establishes the fact that the four properties of a group are satisfied. The operation $\oplus$ is closed, since every cell of the table is occupied by an element of K. The operation $\oplus$ is associative, since it corresponds to the addition of integers which are known to possess this property. The identity element is 0, and each element in K has an inverse. If the inverse of a is designated as a', then $1' = 4$, $2' = 3$, $3' = 2$, $4' = 1$, and $0' = 0$. Thus $\{K: \oplus\}$ is a model possessing group structure.

If $*$ is replaced by $\otimes$ and defined to be the remainder obtained when

the product of any two elements of K is divided by 5, then $\{K: \otimes\}$ is not a group. Group properties G-1, G-2, and G-3 (where the identity element is 1) are satisfied, but G-4 is not, since not every element in K possesses an inverse; $1' = 1$, $2' = 3$, $3' = 2$, $4' = 4$, but 0 has no inverse. If 0 is excluded from consideration, then K becomes the set $\{1,2,3,4\}$, which satisfies the group properties under $\otimes$. The $\otimes$ table is

$\otimes$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Any set K (with zero excluded) which has been formed as a result of using m as a modulus always satisfies group properties G-1 through G-4 under $\otimes$ if and only if m is a prime. This fact is stated without proof. In the example where $K = \{1,2,3,4\}$, $m = 5$. However, if $m = 4$, $K = \{1,2,3\}$ under $\otimes$ is not a group, since the element 2 does not possess an inverse.

Example 7. The familiar set of elements $P = \{1,-1,i,-i\}$ (the fourth roots of unity) forms a group under multiplication where $i = \sqrt{-1}$ and $i^2 = -1$. Here multiplication designated by $\otimes$ is defined as follows:

$\otimes$	1	i	$-i$	-1
1	1	i	$-i$	-1
i	i	-1	1	$-i$
$-i$	$-i$	1	-1	i
-1	-1	$-i$	i	1

This table indicates that G-1 and G-2 are satisfied. The property G-3 holds, since the identity element is 1. Further, each element a has a unique inverse a'; that is, $1' = 1$, $(-1)' = -1$, $i' = -i$, and $(-i)' = i$.

It is interesting to note that this model $P = \{1,-1,i,-i\}$ of a group under $\otimes$ is isomorphic to the model $K = \{1,2,3,4\}$ of a group under $\otimes$ of Example 6. If $1 \leftrightarrow 1$, $-1 \leftrightarrow 4$, $i \leftrightarrow 2$, and $-i \leftrightarrow 3$, then the following illustrates how this correspondence under the operation is maintained for models $\{P: \otimes\}$ and $\{K: \otimes\}$:

Model $\{P: \otimes\}$		Model $\{K: \otimes\}$	Model $\{P: \otimes\}$		Model $\{K: \otimes\}$
-1	$\leftrightarrow$	4	$-i$	$\leftrightarrow$	3
i	$\leftrightarrow$	2	$-i$	$\leftrightarrow$	3
$(-1) \otimes i$	$\leftrightarrow$	$4 \otimes 2$	$(-i) \otimes (-i)$	$\leftrightarrow$	$3 \otimes 3$
$-i$	$\leftrightarrow$	3	-1	$\leftrightarrow$	4

Consequently, the replacement of 1 by 1, i by 2, $-i$ by 3, and -1 by 4 in the table of $\{P: \otimes\}$ yields an exact duplicate of the table for $\{K: \otimes\}$.

Example 8. The motions of a cross may be used to interpret a finite group of four elements. Suppose a cardboard cross is mounted on a sheet of paper upon which are drawn two perpendicular lines, as indicated in Fig. 111. The cross is fastened to the paper at the intersection of the two lines by a paper fastener which permits rotation. It is possible to rotate the cross in either a clockwise or a counterclockwise direction. In order to keep a record of the rotations, the four ends are labeled a, b, c, and d and considered as the initial position for the arrangement indicated in Fig. 111. All rotations take place in integral multiples of 90°.

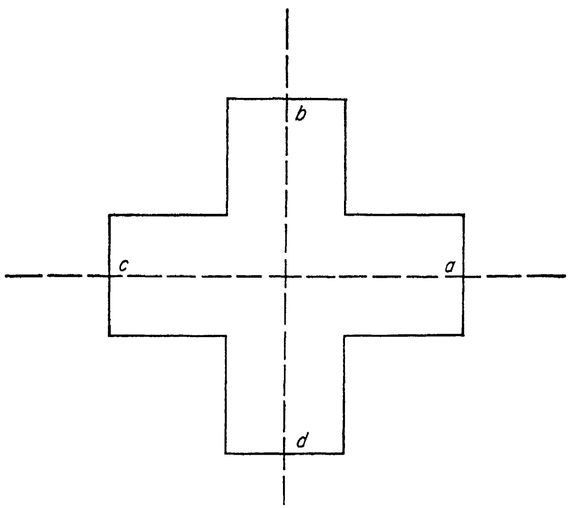


FIG. 111

A counterclockwise rotation of 90° will yield the arrangement indicated in Fig. 112, where the letters placed on the exterior of the cross indicate its previous starting position. This rotation carries the ends a into b, b into c, c into d, and d into a. This arrangement also is representative of a clockwise rotation of 270°.

A counterclockwise (or clockwise) rotation of 180° yields the arrangement in Fig. 113 and carries the ends a into c, b into d, c into a, and d into b.

A rotation of 0° leaves the initial arrangement of the cross unchanged. Note that there exist only four possible rotations that will bring the cross into coincidence with itself.

The following table displays a set of four counterclockwise motions which yield all possible arrangements of the cross under rotation and the

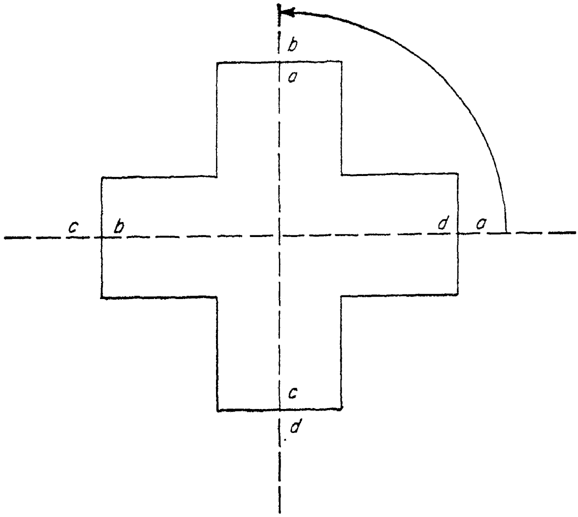


FIG. 112

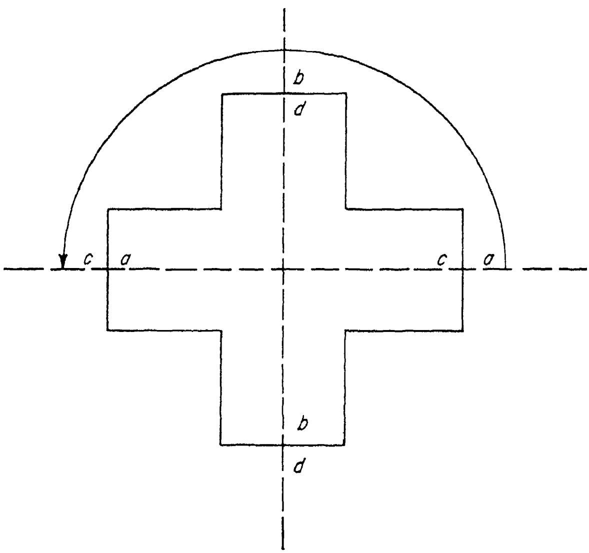


FIG. 113

corresponding symbols used for their representation:

Motion	0°	90°	180°	270°
Symbol	R_0	R_1	R_2	R_3

At this point a set of elements $K = \{R_0,R_1,R_2,R_3\}$ is obtained, each of which represents a motion. If $A,B \in K$, then $A * B$ is defined to mean

the result of the combined rotation where B is performed first and then followed by A. For example, $R_2 * R_3$ means to rotate the cross first through 270° and then follow this with a second rotation of 180°. These rotations in succession produce the same position of the cross as would a single rotation of 90°. Hence $R_2 * R_3 = R_1 = R_3 * R_2$. In terms of a table,

$*$	R_0	R_1	R_2	R_3
R_0	R_0	R_1	R_2	R_3
R_1	R_1	R_2	R_3	R_0
R_2	R_2	R_3	R_0	R_1
R_3	R_3	R_0	R_1	R_2

An examination of the table reveals that the group properties G-1 through G-4 are satisfied. The identity element is R_0 and the inverse elements are $R_0' = R_0$, $R_1' = R_3$, $R_2' = R_2$, and $R_3' = R_1$. Thus $\{K: *\}$ of this example illustrates a group.

Example 9. A rearrangement of the numbers 1, 2, 3, and 4 is called a permutation of these numbers. For example, the arrangement 2341 obtained from 1234 means that 1 is replaced by 2, 2 by 3, 3 by 4, and 4 by 1. Such a rearrangement or permutation is indicated by the element $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, where the top line points out the initial arrangement and the bottom line the final arrangement.

The four distinct permutations, $P_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, $P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, and $P_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$, are now considered as the elements of a set K and are studied from the standpoint of a group. The operation $*$ is defined on the set $K = \{P_0,P_1,P_2,P_3\}$ to mean that if $T,S \in K$, then $T * S$ implies that the permutation S is to be carried out first and then followed by the permutation T on S. For example, $P_3 * P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ can be performed as follows:

$$\begin{array}{c} \quad P_1 \quad P_3 \\ 1 \to 2 \to 1 \\ 2 \to 3 \to 2 \\ 3 \to 4 \to 3 \\ 4 \to 1 \to 4 \end{array}$$

Thus $P_3 * P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, which is the same as P_0, since $1 \to 1$, $2 \to 2$, $3 \to 3$, and $4 \to 4$. Hence the result obtained from the two permutations P_1 and P_3 under $*$ is equivalent to the single permutation P_0.

The following table describes the operation of $*$ on this set K.

$*$	P_0	P_1	P_2	P_3
P_0	P_0	P_1	P_2	P_3
P_1	P_1	P_2	P_3	P_0
P_2	P_2	P_3	P_0	P_1
P_3	P_3	P_0	P_1	P_2

It can be shown that the properties G-1 through G-4 will hold for this set K under $*$. Thus $\{K: *\}$ is a group.

If the ends of the cross designated as a, b, c, and d in Fig. 111 are relabeled 1, 2, 3, and 4, respectively, the permutations P_0, P_1, P_2, and P_3 play the same role as the rotations R_0, R_1, R_2, and R_3 which were discussed in Example 8. It is left as an exercise to verify that the groups of Examples 7 to 9 are isomorphic to one another.

The discussion has been limited to commutative groups, even though this is not a necessary property of a group. This was done so that less complicated examples could be utilized to illustrate group properties. The various models of a group structure point up the importance of the group concept itself. It is apparent that many interpretations can be given to both the elements of a set and the binary operation that help to create a group. The various examples which have been discussed also point out how different interpretations or models of a group may be isomorphic to one another.

Exercise 23

1. Determine in each case whether the given set of elements under the designated operation forms a group.

a. K = set of even integers; $*$ is multiplication.
b. K = set of odd integers; $*$ is multiplication.
c. $K = I$; $*$ is subtraction.
d. K = set of all integral multiples of 3; $*$ is addition.
e. K = set of all positive rational numbers; $*$ is multiplication.
f. K = set of all positive irrational numbers; $*$ is multiplication.
g. K = set of all numbers of the form $a + b\sqrt{2}$, where $a,b \in F$; $*$ is addition.
h. K = set of all polynomials of the form $ax + b$, where $a,b \in I$; $*$ is addition of polynomials.
i. $K = 2^A$, where $A = \{a,b,c\}$; $*$ is union of sets.

2. For Problem 2 of Exercise 20, determine whether the group properties are satisfied for each of the given sets and operations.

3. Given that the motions of an equilateral triangle are:

R_0: No rotation of the triangle
R_1: 120° clockwise rotation of the triangle about its center
R_2: 240° clockwise rotation of the triangle about its center

To visualize these motions, cut out an equilateral triangle from cardboard and then outline this triangle on a sheet of paper along with the additional lines L_1, L_2, and L_3.

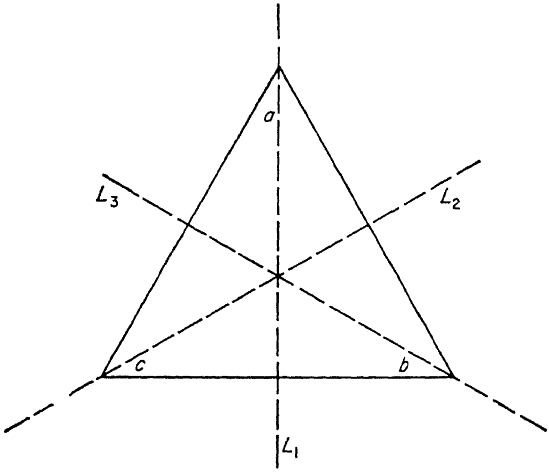
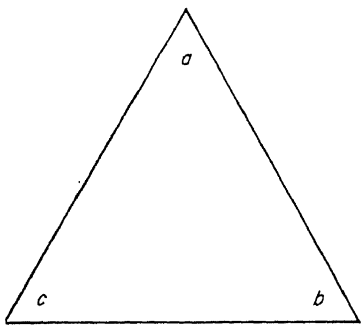


FIG. 114



Cardboard triangle

FIG. 115

Let the vertices of the cardboard triangle be labeled both back and front with the three letters a, b, and c as indicated in Fig. 115. The motion R_1 is produced by placing the cardboard triangle on the outline of Fig. 114 and rotating the cardboard 120°. The rotation R_1 carries the vertex a into b, b into c, and c into a. The rotation R_2 carries a into c, b into a, and c into b, while the rotation R_0 leaves the vertices unchanged. These motions may be designated by the representative figures shown in Fig. 116, where the letters outside the triangle indicate the initial position. If K is considered to be the set $\{R_0,R_1,R_2\}$ and $R_2 * R_1$ is defined to be the combined motion of performing R_1 first and following this by R_2 (on the result of R_1), then K under $*$ forms a group. For example, the arrangement resulting from the com-

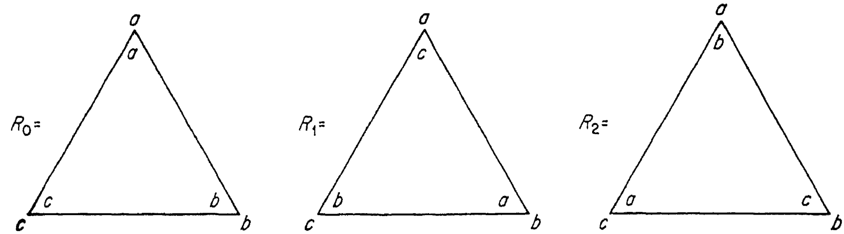


FIG. 116

bined motion of $R_2 * R_1$ can be obtained by using the following scheme:

$$\begin{array}{ccccc} & R_1 & & R_2 & \\ a & \rightarrow & b & \rightarrow & a \\ b & \rightarrow & c & \rightarrow & b \\ c & \rightarrow & a & \rightarrow & c \end{array}$$

Thus $R_2 * R_1 = R_0$, since

$$\begin{array}{c} R_0 \\ a \rightarrow a \\ b \rightarrow b \\ c \rightarrow c \end{array}$$

a. Complete the following table and verify that $K = \{R_0,R_1,R_2\}$ under $*$ forms a group. Is this a commutative group?

$*$	R_0	R_1	R_2
R_0			
R_1			
R_2			

b. If the cardboard triangle of Fig. 115 is lifted and flipped over each of the three lines L_1, L_2, and L_3 of Fig. 114, three new motions of the triangle can be produced that will bring it back into coincidence with itself. These flipping motions are called reflections. Each flipping carries the vertices a, b, and c into different positions. For example, the flipping of the triangle over the line L_1 leaves a unchanged but interchanges b and c. These new motions with respect to the lines L_1, L_2, and L_3 are designated as F_1, F_2, and F_3, respectively, as shown in Fig. 117.

The three reflections now included with the three rotations form the new set $K = \{R_0,R_1,R_2,F_1,F_2,F_3\}$. Though these six motions have been illustrated with

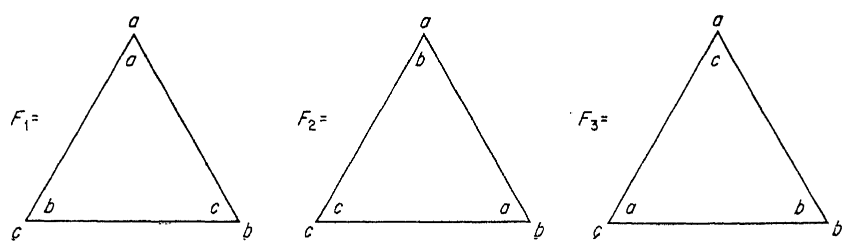


FIG. 117

respect to the designated initial position of the cardboard triangle, it is emphasized that the primary concern is with the motions themselves. The labeling used for the vertices is a device to help keep track of the motions being performed.

The motion $F_1 * R_1$ is defined to mean the combination of the two motions F_1 and R_1, where R_1 is performed first and then followed by F_1 upon the result of R_1.

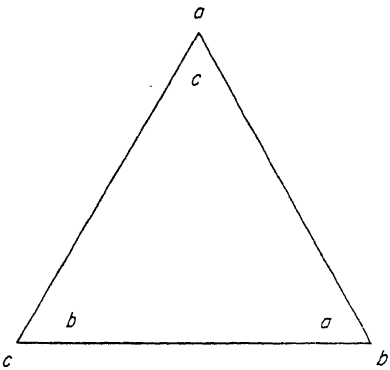


FIG. 118

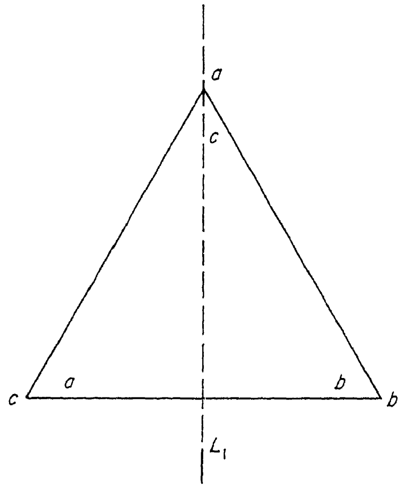


FIG. 119

Thus $F_1 * R_1$ implies that the cardboard triangle is rotated first through 120°, producing the arrangement shown in Fig. 118, and then flipped over the line L_1 with the result shown in Fig. 119. This combined motion can be described as:

$$\begin{array}{c} \quad R_1 \quad F_1 \\ a \to b \to c \\ b \to c \to b \\ c \to a \to a \end{array}$$

The combined motion $F_1 * R_1$ has the same effect as the single motion F_3, and thus $F_1 * R_1 = F_3$.

If the motions to be performed first are listed in the column to the left while those performed second are listed at the top, then the results of the operation $*$ may be summarized in a table.

		Second motion					
	$*$	R_0	R_1	R_2	F_1	F_2	F_3
	R_0	R_0	R_1	R_2	F_1	F_2	F_3
	R_1	R_1	R_2	R_0	F_3	F_1	F_2
First motion	R_2						
	F_1						
	F_2						
	F_3						

Complete this table and verify that $K = \{R_0,R_1,R_2,F_1,F_2,F_3\}$ under $*$ forms a group. Is this a commutative group?

4. There are six possible arrangements of the three numbers 1, 2, and 3. These permutations may be designated as:

$$P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Let $K = \{P_0, P_1, P_2, P_3, P_4, P_5\}$ and $*$ be defined so that $P_4 * P_5$ means to perform P_5 first followed by P_4 on the result of P_5. Thus,

$$\begin{array}{ccccc} P_5 \ \ P_4 & & & & \\ 1 \to 3 \to 2 & \text{or} & 1 \to 2 & & \text{which is } P_1 \\ 2 \to 2 \to 3 & & 2 \to 3 & & \\ 3 \to 1 \to 1 & & 3 \to 1 & & \end{array}$$

It follows that $P_4 * P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = P_1$. Show that $\{K: *\}$ is a group. Is this a commutative group? Is it isomorphic to the group of motions of the equilateral triangle in Problem 3?

5. Is the set of integers modulo 7 a group under $\oplus$? If zero is excluded, will the set of integers modulo 7 be a group under $\otimes$? (See Example 6, Section 5.12.)

5.13 CONCEPT OF A FIELD

A field is a mathematical structure, designated as $\{\mathfrak{F}: \oplus, \otimes\}$, possessing a set of elements $\mathfrak{F}$, two binary operations $\oplus$ and $\otimes$ defined on $\mathfrak{F}$, and satisfying the following postulates:

$\mathfrak{F}$-1: $\{\mathfrak{F}: \oplus\}$ is a commutative group with identity element 0 (zero) and the additive inverse of x denoted by $(-x)$.

$\mathfrak{F}$-2: $\{\mathfrak{F}_0: \otimes\}$ is a commutative group with identity element 1 and the multiplicative inverse of x denoted by $1/x$ or x^{-1}. $\mathfrak{F}_0$ denotes the set of nonzero elements.

$\mathfrak{F}$-3: The operation $\otimes$ is distributive over $\oplus$.

The equivalence relation designated by the equality sign represents the fact that if $a = b$, then a and b are symbols for the same element.

The individual postulates of a field appear in Section 2.7 but are there interpreted with respect to real numbers. However, if these postulates are rewritten with R_e replaced by $\mathfrak{F}$ (a set of abstract elements), then the general concept of a field will be the result. In this manner the properties of a field can be studied as an abstract structure without any reference to a particular model. In Chapter 2 the sets F and R_e, together with the operations of addition and multiplication, were shown to be models of a field. Further, the examples of Chapter 2 illustrated the format to be used for proving theorems. In addition, it was shown that many of the well-known techniques of algebra follow as logical consequences of the postulates of a field.

As an added illustration of the format for proof, the following theorem referred to as the "cancellation law for addition" is included. If $a,b,c \in \mathfrak{F}$ such that $a + b = a + c$, then $b = c$.

Proof:		*Authority*
1.	$a + b = a + c$	1.
2.	$(-a) + (a + b) = (-a) + (a + c)$	2.
3.	$(-a) + a + b = (-a) + a + c$	3.
4.	$0 + b = 0 + c$	4.
5.	$b = c$	5.

The authority or reason for each step is left as an exercise.

The principle of duality is a property possessed by a Boolean algebra and not by a field. However, there are various theorems derived from the field postulates that exhibit interesting parallelisms. As examples,

1. If $a,b,c \in \mathfrak{F}$ and if $a + b = a + c$, then $b = c$.	1. If $a,b,c \in \mathfrak{F}$, $a \neq 0$, and $a \cdot b = a \cdot c$, then $b = c$.
2. If $a,b \in \mathfrak{F}$ and if $a + b = 0$, then $b = -a$.	2. If $a,b \in \mathfrak{F}$, $a \neq 0$, and $a \cdot b = 1$, then $b = 1/a$.
3. If $a \in \mathfrak{F}$, then $a = -(-a)$.	3. If $a \in \mathfrak{F}$, $a \neq 0$, then $a = \dfrac{1}{1/a}$ or $a = (a^{-1})^{-1}$.

Thus, if the theorems at the left are proved, it is possible to prove corresponding theorems at the right by replacing $\oplus$ by $\otimes$, 0 by 1, and $-a$ by $1/a$. The duality principle is not a universal property for a field structure, since parallel statements can be created which are not true of all the elements in $\mathfrak{F}$. For example, $a \otimes 0 = 0$ is true for all elements in $\mathfrak{F}$, but the parallel statement $a \oplus 1 = 1$ does not follow for all elements in $\mathfrak{F}$.

Example 1. The set of integers modulo 5 where $K = \{0,1,2,3,4\}$ is a field. It was shown in Example 6 of Section 5.12 that K is a commutative group under addition and $K_0 = \{1,2,3,4\}$ is a commutative group under multiplication. It can also be shown that $\otimes$ is distributive over $\oplus$.

Example 2. The set of integers modulo 6 where $K = \{0,1,2,3,4,5\}$ is not a field. The $\oplus$ table for K and the $\otimes$ table for K_0 are

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\otimes$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

K is a commutative group under $\oplus$, but K_0 fails to be such under $\otimes$. The table exposes the fact that the elements 2, 3, and 4 do not have multiplicative inverses.

Example 3. The set K of numbers of the form $a + b\sqrt{2}$, where $a,b \in F$, is a field. To show that K is a commutative group under addition:

G-1: $(x + y\sqrt{2}) + (u + v\sqrt{2}) = (x + u) + (y + v)\sqrt{2}$. If $x,y,u,v \in F$, then $(x + u) \in F$ and $(y + v) \in F$. It follows that $(x + u) + (y + v)\sqrt{2}$ is of the form $a + b\sqrt{2}$.

G-2: Since associativity holds for real numbers, it will hold for real numbers of the form $a + b\sqrt{2}$.

G-3: $0 + 0 \cdot \sqrt{2} = 0$ is the identity element for addition.

G-4: $-(a + b\sqrt{2})$ is the additive inverse of $a + b\sqrt{2}$, since $(a + b\sqrt{2}) + [-(a + b\sqrt{2})] = 0 + 0 \cdot \sqrt{2} = 0$.

G-5: Since commutativity holds for real numbers, it will hold for real numbers of the form $a + b\sqrt{2}$.

In a similar manner it can be shown that the set of numbers of the form $a + b\sqrt{2}$ (0 excluded) is a commutative group under multiplication with identity element $1 + 0 \cdot \sqrt{2} = 1$. The multiplicative inverse of

$$a + b\sqrt{2} \text{ is } \frac{1}{a + b\sqrt{2}} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \qquad (a^2 - 2b^2 \neq 0)$$

It is left as an exercise to show that multiplication is distributive over addition.

Example 4. The set of complex numbers of the form $a + bi$, where $a,b \in R_e$, $i = \sqrt{-1}$, $i^2 = -1$, is another model of a field. The verification of the postulates as they apply to numbers of the form $a + bi$ is left as an exercise.

5.14 CONCEPT OF AN ORDERED FIELD

If a field satisfies an additional set of postulates that imposes an order relation upon its elements, then the field is called an "ordered field." In this case it is assumed that the field possesses in addition to the equivalence relation of equality a second relation denoted by the symbol "$<$" and read "less than." The statement "$a < b$" is read "a less than b" and, by agreement, means the same as "$b > a$" which is read "b greater than a." Thus if $\{\mathfrak{F}: \oplus, \otimes\}$ is an ordered field, it possesses

the order relation "$<$" and satisfies the following "order postulates":

P-1: If $a,b \in \mathfrak{F}$, then one and only one of the following is true: $a < b$, $a = b$, $a > b$. (Law of trichotomy)
P-2: If $a,b,c \in \mathfrak{F}$ and $a < b$, then $a + c < b + c$.
P-3: If $a,b,c \in \mathfrak{F}$ and if $c > 0$ and $a < b$, then $ac < bc$.
P-4: If $a,b,c \in \mathfrak{F}$ such that $a < b$ and $b < c$, then $a < c$.

By convention, it is agreed to say that "a is positive" when "$a > 0$" or that "a is negative" when "$a < 0$."

The two fields $\{F: +,\times\}$ and $\{R_e: +,\times\}$ are ordered while the fields created from I (integers) modulo p (a prime) are not ordered. It is also noted that the field $\{C: +,\times\}$, where C is the set of complex numbers, is not ordered.

Many basic theorems result as a consequence of the added order postulates for a field. As illustrations, a number of these theorems are proved.

Order Theorem 1. If $a,b,c \in \mathfrak{F}$ and if $a < b$ and $c < d$, then $a + c < b + d$.

Proof:	*Authority*
1. $a < b$	1. By hypothesis
2. $a + c < b + c$	2. P-2
3. $c < d$	3. By hypothesis
4. $b + c < b + d$	4. P-2
5. $a + c < b + d$	5. P-4

Order Theorem 2. If $a \in \mathfrak{F}$ and $a < 0$, then $-a > 0$.

Proof:	*Authority*
1. $a < 0$	1. By hypothesis
2. $(-a) + a < (-a) + 0$	2. P-2
3. $0 < -a$ or $-a > 0$	3. Field postulates with respect to additive inverse and identity elements

Order Theorem 3. If $a,b \in \mathfrak{F}$ and $a < b$, then $-a > -b$.

Proof:	*Authority*
1. $a < b$	1.
2. $(-a) + a < (-a) + b$	2.
3. $0 < (-a) + b$	3.
4. $0 + (-b) < (-a) + b + (-b)$	4.
5. $-b < -a + 0$	5.
6. $-b < -a$ or $-a > -b$	6.

The reasons for the various steps in Order Theorem 3 are omitted and left as an exercise.

Order Theorem 4. If $a,b,c \in \mathfrak{F}$, and if $a < b$ and $c < 0$, then $ac > bc$.

Proof:	*Authority*
1. $a < b$	1. By hypothesis
2. $c < 0$	2. By hypothesis
3. $-c > 0$	3. Order Theorem 2
4. $-ac < -bc$	4. P-2
5. $ac > bc$	5. Order Theorem 3

Order Theorem 5. If $a,b \in \mathfrak{F}$, then $a < b$ if and only if $b - a > 0$.
a. If $a < b$, then $b - a > 0$.

Proof:	*Authority*
1. $a < b$ or $b > a$	1. By hypothesis
2. $b + (-a) > a + (-a)$	2. P-2
3. $b + (-a) > 0$	3. Since $a + (-a) = 0$
4. $b - a > 0$	4. By definition

b. If $b - a > 0$, then $a < b$.
The proof of part *b* is left as an exercise.

Exercise 24

1. Determine whether the given set of elements under the operations of addition and multiplication forms a field.
a. K = set of rational numbers
b. K = set of integers
c. K = set of real numbers
d. K = set of integers modulo 3
e. K = set of integers modulo 4
f. K = set of integers modulo 9
g. K = set of numbers of the form $a + b\sqrt{2}$ where $a,b \in I$
h. K = set of numbers of the form $a + b\sqrt{3}$ where $a,b \in F$
i. K = set of complex numbers of the form $a + bi$ where $a,b \in R_s$

2. Prove the following theorems for an ordered field:
a. If $a,b \in \mathfrak{F}$, and if $a > 0$, and $b > 0$, then $a + b > 0$ and $ab > 0$.
b. If $a,b \in \mathfrak{F}$, and if $a > 0$ and $b < 0$, then $ab < 0$.
c. If $a,b \in \mathfrak{F}$, and if $a < 0$ and $b < 0$, then $a + b < 0$ and $ab > 0$.
d. If $a,b,c \in \mathfrak{F}$ and if $ac < bc$ and $c > 0$, then $a < b$.
e. If $a,b,c \in \mathfrak{F}$ and if $c < 0$, then $a < b$ if and only if $ac > bc$.
f. If $a \in \mathfrak{F}$ and $a \neq 0$, then $a^2 > 0$.
g. If $a \in \mathfrak{F}$, then $a > 0$ if and only if $a^{-1} > 0$.
h. If $a,b \in \mathfrak{F}$ and if $a > b$ and $ab > 0$, then $a^{-1} < b^{-1}$.

PROJECTS

Supplementary Exercises

1. If A is a subset of a universal set U, then the complement of A, written A', is defined to be that set of all elements of U not contained in A. Frequently only a part of the complement of a set may be of interest. For example, if A and B are two subsets of U, then it might be of concern to investigate that part of the complement of B which is contained in A. This set, described as $\{x \mid x \in A \wedge x \notin B\}$, is referred to as the "difference of the two subsets A and B" and is designated $A - B$. Thus

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

or

$$A - B = A \cap B'$$

a. Draw a Venn diagram illustrating the set $A - B$ $(A \cap B \neq \emptyset)$.

b. The complement of a subset is considered to be a special case of a difference set; that is, if A is a subset of U, then $A' = U - A$. Complete the following statements:

(1) $A - U =$	(2) $\emptyset - A =$	(3) $A - \emptyset =$
(4) $A - A =$	(5) $U - \emptyset =$	(6) $\emptyset - U =$

Example. $A - U = A \cap U' = A \cap \emptyset = \emptyset$

c. Using the definition of "difference" and the laws of the algebra of sets, prove the following:

(1) $(A - B) - C = A - (B \cup C)$
(2) $A \cup (B - C) = (A \cup B) - (C - A)$
(3) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(4) $(A - B)' = A' \cup B$
(5) $(A \cup B) - C = (A - C) \cup (B - C)$
(6) $(A \cap B) - C = (A - C) \cap (B - C)$
(7) $A - (B - C) = (A - B) \cup (A \cap C)$

d. Is set difference a commutative operation?

2. The symmetric difference $A \triangle B$ of two elements A and B of a Boolean algebra is defined by $A \triangle B = AB' + A'B$ or, equivalently, $A \triangle B = (A + B)(A' + B')$.

a. Show that $AB' + A'B = (A + B)(A' + B')$.

b. Suppose A and B are sets. Rewrite the above definition for "$\triangle$" by use of the symbols $\cup$ and $\cap$. Construct a membership table for $A \triangle B$. From a Venn diagram, identify the region or regions represented by $A \triangle B$. Describe the set $A \triangle B$ in words.

c. By use of either membership tables or the postulates of the algebra of sets, verify each of the following laws of "$\triangle$":

(1) $A \triangle \emptyset = A$	(2) $A \triangle U = A'$
(3) $A \triangle A = \emptyset$	(4) $A \triangle A' = U$
(5) $A \triangle B = B \triangle A$	(6) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$

d. Using the definition of "symmetric difference" and the postulates of the algebra of sets, prove the following:

(1) $(A \triangle B)' = (A' \cup B) \cap (A \cup B')$
(2) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

e. Interpret $A \triangle B$ with respect to the Boolean algebra of the integral divisors of 6 by using the definitions of $+$, $\cdot$, and $'$ as given in Example 2, Section 5.8. Construct its corresponding table for $A \triangle B$.

3. Design an electric circuit which simulates the moves of the following puzzle: A father and two sons wish to cross a river. The father weighs 150 pounds and each son weighs 75 pounds. The only available boat is capable of carrying 150 pounds. How do they all succeed in getting across?

4. A hall light is controlled by three switches. Design an electric circuit which effectively controls the hall light so that the throwing of any one of the switches turns the light on or off.

5. The mathematical structure called a "ring" consists of a nonempty set of elements K on which are defined two binary operations called addition $\oplus$ and multiplication $\otimes$ with the following properties satisfied:

(1) $\{K: \oplus\}$ is a commutative group.

(2) Closure and associativity hold for $\otimes$.

(3) $\otimes$ is distributive over $\oplus$. Note that commutativity does not have to hold for $\otimes$. Therefore the distributive law is stated in two parts:

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$
$$(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$$

Determine whether each of the indicated sets under the operations of addition and multiplication forms a ring:

a. K = set of integers

b. K = set of even integers or $K = \{. . . , -4, -2, 0, 2, 4, . . .\}$

c. K = set of all numbers of the form $a + b\sqrt{2}$, where $a,b \in I$

d. K = set of integers modulo 4

e. K = set of integers modulo 3

f. K = set of integers with the following new definitions of "addition" and "multiplication": If $a,b \in I$, then $a \oplus b = a + b - 1$ and $a \otimes b = a + b - ab$.

6. The statement $[(p \to q) \wedge p] \to q$ is a tautology and is referred to as the law of inference or law of detachment, since the conclusion q may be inferred or detached from the compound statement $[(p \to q) \wedge p]$. This law is frequently used in mathematics along with other tautologies in the development of a deductive proof. If we write the tautology $[(p \to q) \wedge p] \to q$ in the form $\begin{array}{c} p \to q \\ \underline{p \quad\quad} \\ q \end{array}$, the following examples illustrate its use.

Example 1:

$p \to q$:	If I pass this course, then I shall graduate.
p:	I passed this course.
q:	I shall graduate.

Example 2: (euclidean plane geometry)

$p \to q$:	If two lines a and b are not perpendicular to a third line c, then they are parallel.
p:	a and b are not perpendicular to c.
q:	a is parallel to b.

In Examples 1 and 2 we asserted $p \to q$ (the major premise) and p (the minor premise) without any concern for the truth values assigned to either p or $p \to q$. Our major interest is in the symbolic form of the compound statement $[(p \to q) \wedge p] \to q$, which is a tautology; i.e., this law is valid and is independent of the truth values assigned to its component parts.

In each of the following, determine whether the conclusion is correctly inferred from its major and minor premises (no judgment is to be made relative to the truth or falsity of the premises).

a. Major premise: If it snows, then I shall go swimming.
Minor premise: It is snowing.
Conclusion: I shall go swimming.

b. Major premise: If a triangle is isosceles, then its base angles are equal.
Minor premise: Triangle GHI is isosceles ($GH = HI$).
Conclusion: Angle G is equal to angle I.

7. Consider the following argument:

Major premise: If two angles are right angles, then they are equal.
Minor premise: Angle A = angle B.
Conclusion: A and B are right angles.

8. Consider:

p: This month is May.
q: Last month was April.
r: Six months ago it was June.

Analyze the following statement:

If this month is May, then last month was April. But if last month was April, then six months ago it was June. Then if this month is May, six months ago it was June.

Does the conclusion logically follow? (*Hint:* Prove that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.)

Answers to Selected Problems

CHAPTER 1

Exercise 1, Page 3

1*a*. Not a well-defined set
b. Well-defined set
c. Not a well-defined set

2*a*. $r \in Q$
d. $h \notin Q$
f. $g \notin Q$

Exercise 2, Page 4

3*a*. Finite set; eight elements
b. Infinite set
d. Finite set; two elements
f. Infinite set
h. Finite set; 0 elements

Exercise 3, Page 7

1*a*. Finite set; four elements
$\{M,i,s,p\}$
b. Infinite set
f. Finite set; nine elements
$\{7,11,13,17,19,23,29,31,37\}$
i. Infinite set

2*a*. x is greater than -2 but less than 8
$H = \{-1,0,1,2,3,4,5,6,7\}$
d. x has less than 30 days
$A = \{\text{February}\}$
g. $x + 1 = x$
$G = \{\ \ \} = \emptyset$
h. $x + 2x + 5 = 3x + 5$
$K =$ set of rational numbers
k. $2x - 5$ is less than 6
$Q = \{1,2,3,4,5\}$

4.

Statement	N	I	F	R_e
a. $3x - 4 = 2$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
c. $x^2 = 4$	$\{2\}$	$\{2,-2\}$	$\{2,-2\}$	$\{2,-2\}$
e. $x^2 = 5$	$\emptyset$	$\emptyset$	$\emptyset$	$\{\sqrt{5},-\sqrt{5}\}$
i. $x^2 - 3x = 0$	$\{3\}$	$\{0,3\}$	$\{0,3\}$	$\{0,3\}$

6*a*. True
d. False
g. False
h. True

Exercise 4, Page 10

1.
$$\begin{matrix} 1 & 2 & 3 \\ \updownarrow & \updownarrow & \updownarrow \\ a & b & c \end{matrix}$$
Six ways

3.
$$\begin{matrix} 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1 & 3 & 5 & 7 & \cdots & 2n-1 & \cdots \end{matrix}$$

5*a*. 1; 3; 7

Exercise 5, Page 12

1*b*. $A = B$
d. $A = B$
f. $A \neq B$
g. $A \neq B$

Exercise 6, Page 17

1*a*. {Tuesday, Thursday}
{$*$ | $*$ is a day of the week beginning with the letter T}
{Tuesday, Thursday} $\subseteq$ {all days of the week}
c. $\{5,6\}$
{x | x is a natural number between 4 and 7}
$\{5,6\} \subseteq$ {x | x is a natural number greater than 4}

3. $N \subset R_e$ $\quad R_i \subset R_e$
$N \subset F$ $\quad I \subset R_e$
$N \subset I$ $\quad I \subset F$
$F \subset R_e$ $\quad H \subset R_e$
$R_e^- \subset R_e$ $\quad H \subset N$
$I^+ \subset R_e$ $\quad H \subset F$
$I^+ \subset F$ $\quad H \subset I^+$
$I^+ \subset I$ $\quad H \subset I$

4*a*. True
d. True
g. False
k. False

5*a*. $\{10\}$
d. I
e. $\emptyset$
h. $\{0\}$

6*a*. $\{3\}$
d. $\{\sqrt{3}, -\sqrt{3}\}$
h. $\emptyset$
i. $\{x \mid x \in R_e \text{ and } x \neq 0\}$
n. $\{0,4\}$

8*b*. Yes
d. No
e. Yes

Exercise 7, Page 20

1*f*. $\{\{0,\{0\},\emptyset,\{\emptyset\}\};\{0,\{0\},\emptyset\};\{0,\{0\},\{\emptyset\}\};\{0,\emptyset,\{\emptyset\}\};\{\{0\},\emptyset,\{\emptyset\}\};\{0,\{0\}\};$
$\{0,\emptyset\};\{0,\{\emptyset\}\};\{\{0\},\emptyset\};\{\{0\},\{\emptyset\}\};\{\emptyset,\{\emptyset\}\};\{0\};\{\{0\}\};\{\emptyset\};\{\{\emptyset\}\};\{\emptyset\}\}$
All but $\{0,\{0\},\emptyset,\{\emptyset\}\}$ and $\emptyset$ are proper subsets; 16 subsets

3. $2^n - 1$; $2^n - 2$;
No, the given set of n elements is an element of the set of all nonempty subsets but is not contained in the set of all proper subsets.

4*a*. False
d. False
f. True
i. True
j. False
m. False

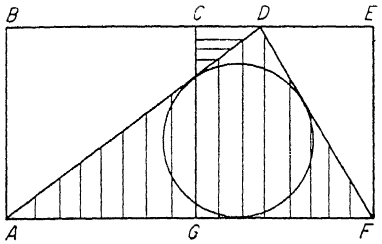
6*a*. True
c. False
Let $T = \{1\}$
$S = \{\{1\},2\}$
$V = \{\{\{1\},2\},5\}$
$T \in S$, $S \in V$, but $T \notin V$

Exercise 8, Page 23

2*a*. Set of even natural numbers
b. Set of positive integers and zero

4*a*. $A \cap B = \emptyset$
c. $A' = B$
f. $B \cap C = \{10,20,30, \ldots\}$
h. $A \cap B \cap \emptyset = \emptyset$

6*a*. $A \cap B = \emptyset$
$A \cup B = \{\text{all the elements of } A \text{ or } B\}$
c. $A \cap B = A$
$A \cup B = B$

7 

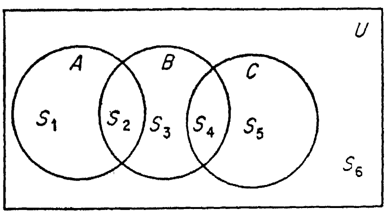
a. $L \cap M$ ||||
d. $M' \cap W'$ ≡
h. $M' \cap T = \emptyset$
j. M'
m. $W' \cap (M' \cup L)$

Exercise 9, Page 34

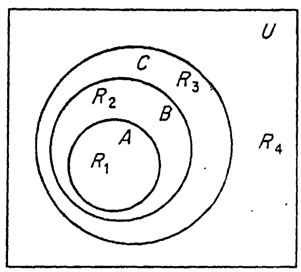
2*a*. $A - B = \{2,6,10,14,18\}$
d. $(B - A)' = U$
h. $(B - A)' \cap (A - B)' = \{1,3,4,5,7,8,9,11,12,13,15,16,17,19\}$

3*e*.

$A \cap (B \cup C)$		$(A \cap B) \cup (A \cap C)$	
Set	Regions	Set	Regions
A	S_1, S_2, S_3, S_4	$A \cap B$	S_1, S_2
$B \cup C$	$S_1, S_2, S_3, S_4, S_5, S_6, S_7$	$A \cap C$	S_1, S_3
$A \cap (B \cup C)$	S_1, S_2, S_3	$(A \cap B) \cup (A \cap C)$	S_1, S_2, S_3

4*b*. 

Note: This is not the only Venn diagram that will satisfy the given conditions.

f. 

5.

Set	Venn diagrams of 4*b*	Venn diagrams of 4*f*
a. $A \cap (B \cap C)$	$\emptyset$	R_1
b. $(A \cap B) \cup C$	S_2, S_4, S_5	R_1, R_2, R_3
c. $(A \cup B) \cap C$	S_4	R_1, R_2
d. $A \cup (B \cap C)$	S_1, S_2, S_4	R_1, R_2
e. $A \cap (B \cup C)$	S_2	R_1

8*a*. 34
b. 10
c. 5
d. 9

11*a*. 55 per cent
c. 65 per cent
e. 25 per cent

CHAPTER 2

Exercise 10, Page 58

2*a*. False
c. False
d. False
i. True
k. True
m. True

4*a*. $-2, \frac{1}{2}$
c. $-\frac{1}{2}, 2$
e. $-\frac{3}{4}, \frac{4}{3}$
g. 0, no multiplicative inverse
i. $3\frac{1}{4}, -\frac{4}{13}$
k. $\sqrt{3}, -\frac{1}{\sqrt{3}}$
m. $3 - x, \frac{1}{x-3} \quad (x \neq 3)$
o. $1 - \frac{2}{x}, \frac{x}{2-x} \quad (x \neq 2)$

5.

$+$	$-$	$\cdot$	$\div$
Yes	No	Yes	No
Yes	No	No	No
Yes	No	Yes	No
Yes	No	No	No
No	No	No	No
Yes	No	Yes	No
No	No	Yes	No

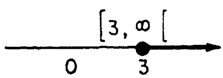
8*a*. $\frac{56}{99}$
c. $\frac{5678}{9999}$
e. $\frac{227}{165}$
g. $\frac{119}{55}$

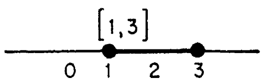
9*a*. 2.00000 · · ·
c. 1.29099 · · ·
e. 1.00000 · · ·
g. 0.22222 · · ·
i. 2.66666 · · ·
l. 2.89523 · · ·
o. 0.26794 · · ·
q. 4.24264 · · ·
u. −0.17157 · ·

Exercise 11, Page 70

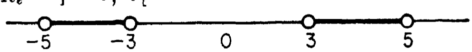
1*a*. False
c. True
e. True
g. True
i. False
k. True

3*a*. 4
c. 30
e. −2
g. −16

5*a*.


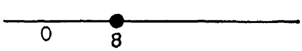
c.


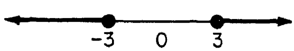
e. $R_e =]-\infty, \infty[$

g.


$]-5,-3[\cup]3,5[$

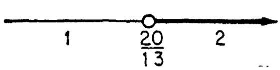
6*a*. [2,11]
c. [−3,1]
d.]7,9]
f. ∅
h.]−3,5[
j. [2,3]

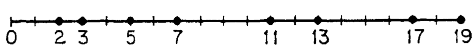
8*a*.


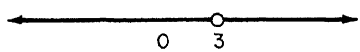
c.


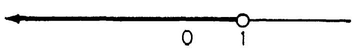
e.


g.


i.


k.


m.


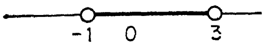
o.


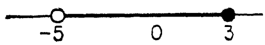
q. ∅

s.

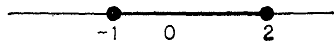

9*a*. $\emptyset$ *e*. $\{3\}$
c. $\{0,1\}$ *g*. $\{0,1,2\}$

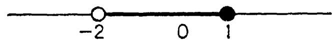
10*a*. $\{x \mid x \geq 5\}$ *g*. $\{x \mid x \notin]1,3[\}$
c. $\{x \mid x < -22\}$ *i*. $\{2, -\frac{2}{3}\}$
e. $\{x \mid x \notin [-\frac{1}{2},3]\}$ *k*. $\emptyset$

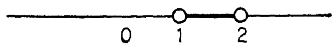
12*a*. 

c. 

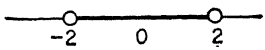
e. $\emptyset$

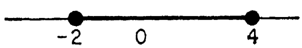
14*a*. 

c. 

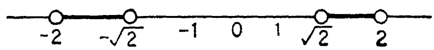
e. 

15*a*. $\emptyset$

c. 

e. 

g. $\emptyset$

i. 

CHAPTER 3

Exercise 12, Page 78

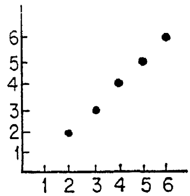
1*a*. $A \times A = \{(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$
d. $B \times A = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5)\}$
f. $(A \times A) \cap (B \times B) = \{(3,3),(3,4),(4,3),(4,4)\}$
i. $(A \times B) \cap (B \times A) = \{(3,3),(3,4),(4,3),(4,4)\}$

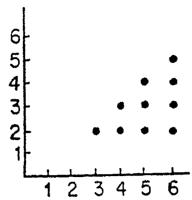
2*a*. n^2 elements
c. mn elements
e. n elements

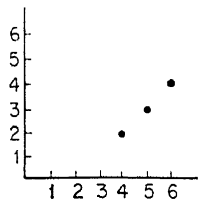
3. $I^+ \times N = N \times I^+$

Exercise 13, Page 83

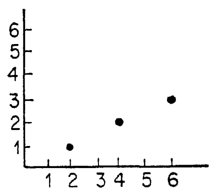
1*a*. $\{(2,2),(3,3),(4,4),(5,5),(6,6)\}$
$\{(x,y) \in U \times U \mid x = y\}$
$D^* = \{2,3,4,5,6\}$
$R^* = \{2,3,4,5,6\}$
e. $\{(3,2),(4,2),(4,3),(5,2),(5,3),(5,4),(6,2),(6,3),(6,4),(6,5)\}$
$\{(x,y) \in U \times U \mid x > y\}$
$D^* = \{3,4,5,6\}$
$R^* = \{2,3,4,5\}$
h. $\{(4,2),(5,3),(6,4)\}$
$\{(x,y) \in U \times U \mid x = y + 2\}$
$D^* = \{4,5,6\}$
$R^* = \{2,3,4\}$

2*a*.


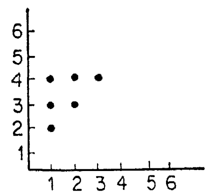
e.


h.


3*a*. No
b. Yes
d. No
f. No
l. No
n. Yes

4*b*.


$D^* = \{2,4,6\}$
$R^* = \{1,2,3\}$

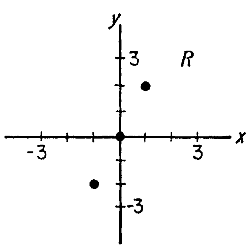
f.


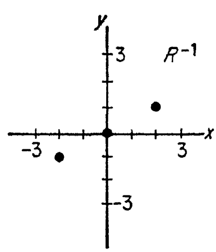
$D^* = \{1,2,3\}$
$R^* = \{2,3,4\}$

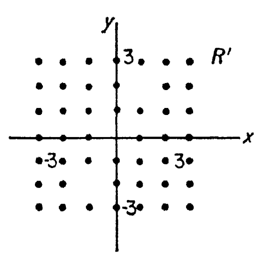
	Relation	*Universe*
6*a*.	"is as wealthy as"	All people
b.	"is acquainted with"	All people
c.	"is the father of"	All people
d.	"is poorer than"	All people
e.	"is a multiple of"	Natural numbers

Exercise 14, Page 88

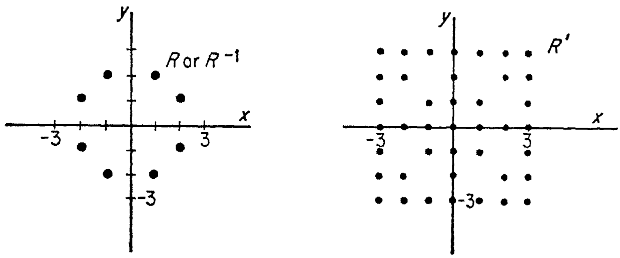
1*a*. $R = \{(-1,-2),(0,0),(1,2)\}$
$D^* = \{-1,0,1\},\ R^* = \{-2,0,2\}$
$R^{-1} = \{(-2,-1),(0,0),(2,1)\}$
$D^* = \{-2,0,2\},\ R^* = \{-1,0,1\}$
$R' = \{(-2,-2),(-2,-1),(-2,0),(-2,1),(-2,2),(-2,3),(-1,-1),(-1,0),$
$(-1,1),(-1,2),(-1,3),(0,-2),(0,-1),(0,1),(0,2),(0,3),(1,-2),$
$(1,-1),(1,0),(1,1),(1,3),(2,-2),(2,-1),(2,0),(2,1),(2,2),(2,3),(3,-2),$
$(3,-1),(3,0),(3,1),(3,2),(3,3)\}$
$D^* = \{-2,-1,0,1,2,3\},\ R^* = \{-2,-1,0,1,2,3\}$







f.



2*a*. $R^{-1} = \{(x,y) \mid x = 2y\}$, $R \neq R^{-1}$

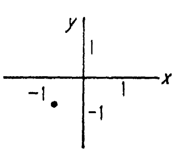
c. $R^{-1} = \{(x,y) \mid x = 3y - 3\}$, $R \neq R^{-1}$

f. $R^{-1} = \{(x,y) \mid x^2 + y^2 = 5\}$, $R = R^{-1}$

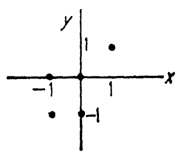
3.

Relation	Complement	Inverse
$=$	$\neq$	$=$
$\subset$	$\not\subset$	$\supset$
"is married to"	"is not married to"	"is married to"

5*a*.

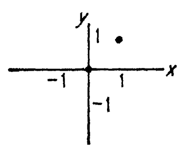


$R_1 \cap R_2$

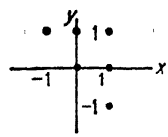


$R_1 \cup R_2$

d.



$R_1 \cap R_2$



$R_1 \cup R_2$

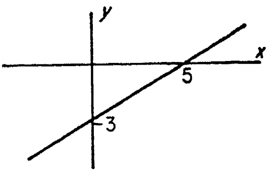
7*a*.

$U \times U$	R_1	R_2	R_3	$R_2 \cap R_3$	$R_1 \cup (R_2 \cap R_3)$	$R_1 \cup R_2$	$R_1 \cup R_3$	$(R_1 \cup R_2) \cap (R_1 \cup R_3)$
(1,1)	$\notin$	$\in$	$\notin$	$\notin$	$\notin$	$\in$	$\notin$	$\notin$
(1,2)	$\in$	$\notin$	$\in$	$\notin$	$\in$	$\in$	$\in$	$\in$
(1,3)	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
(2,1)	$\in$	$\notin$	$\in$	$\notin$	$\in$	$\in$	$\in$	$\in$
(2,2)	$\notin$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
(2,3)	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
(3,1)	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
(3,2)	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
(3,3)	$\notin$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$

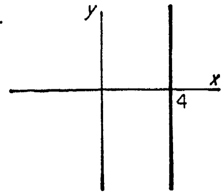
CHAPTER 4

Exercise 15, Page 115

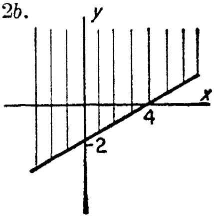
1*b*.



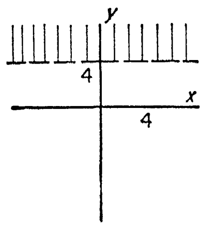
f.



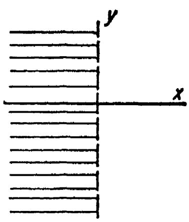
2*b*.



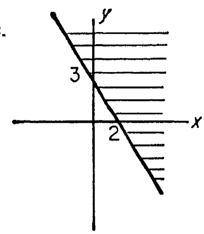
d.



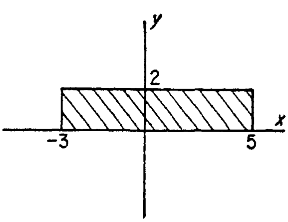
g.



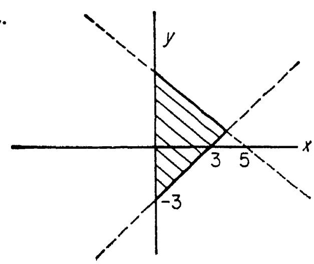
k.



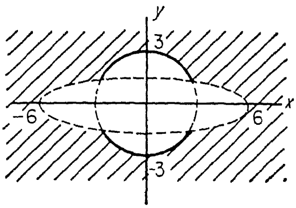
4*c*.



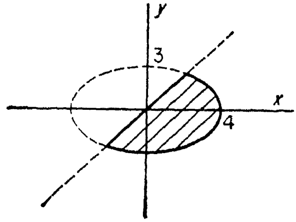
h.



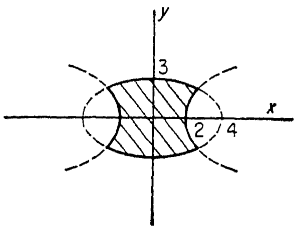
n.



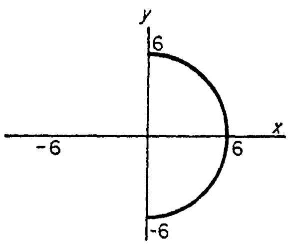
q.



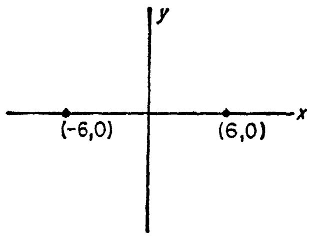
u.



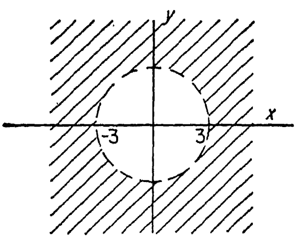
6*a.*



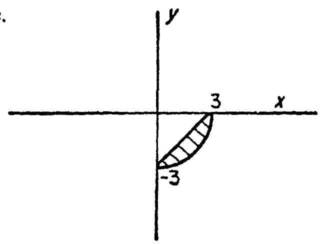
c.



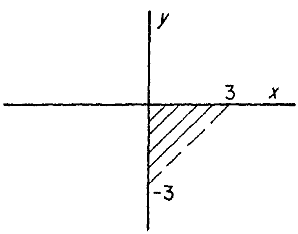
7*b.*



e.

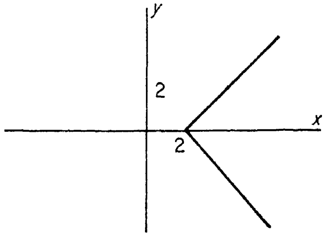


i.

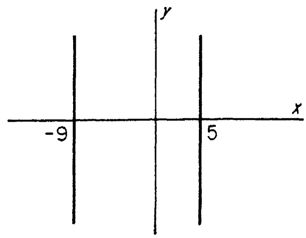


Exercise 16, Page 119

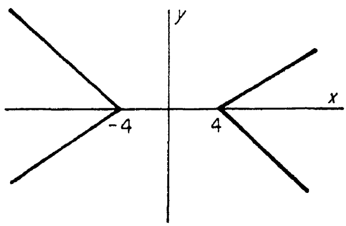
1*a*.



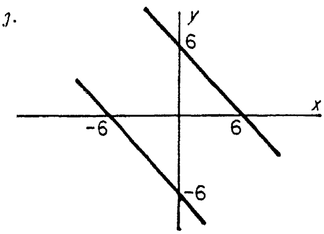
c.



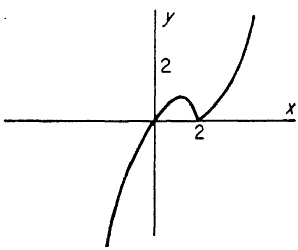
e.



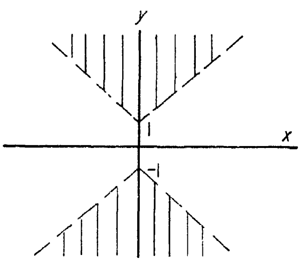
j.



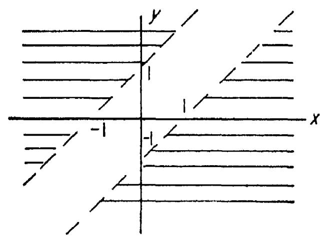
k.



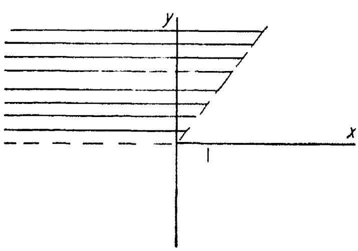
2*a*.



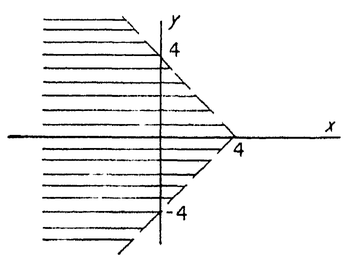
c.



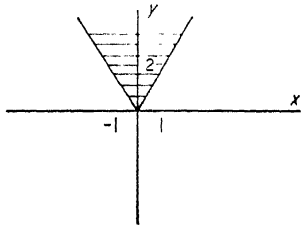
f.



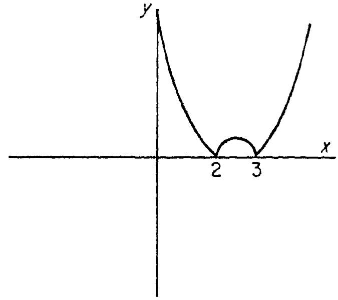
i.



3*b*.



4*a*.



Exercise 17, Page 127

1*a*. Yes

c. No

e. No

2*a*. Yes

3*a*. $f = \{(x,f(x)) \mid f(x) = \sqrt{4 - x^2}\}$

d. $f = \{(x,f(x)) \mid f(x) = 0\}$

h. $f = \{(x,f(x)) \mid f(x) = -x - 1\}$

4*a*. $f = \{(4,13),(1,-2),(2,1),(0,3)\}$

d. $f = \{(x,y) \mid y = x^3 \wedge x \in]0, \infty[\}$

7*a*. $D^* = [\frac{1}{4}, \infty[$; $R^* = [0, \infty[$

d. $D^* = [-2, \infty[$; $R^* = [0, \infty[$

10*a*. $f = \{(r,V) \mid V = f(r) = \frac{4}{3}\pi r^3 \wedge r \in]0, \infty[\}$

c. $f = \{(w,V) \mid V = f(w) = 2w(w - 4)^2 \wedge w \in]4, \infty[\}$

e. $f = \{(s,A) \mid A = f(s) = \dfrac{s^2\sqrt{3}}{4} \wedge s \in]0, \infty[\}$

g. $f = \{(r,A) \mid A = f(r) = 4\pi r^2 \wedge r \in]0, \infty[\}$

j. $f = \{(r,V) \mid V = f(r) = \dfrac{28\pi r^3}{3} \wedge r \in]0, \infty[\}$

12. $f = \left\{(x,A) \mid A = \dfrac{x(200 - x)}{2} \wedge x \in]0,200[\right\}$

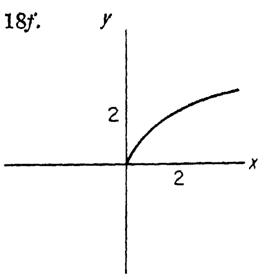
14. $f = \left\{(x,A) \mid A = 2x\left(\dfrac{100 - 2x}{2 + \pi}\right) + \dfrac{\pi}{2}\left(\dfrac{100 - 2x}{2 + \pi}\right)^2 \wedge x \in]0,50[\right\}$

16. $f = \left\{(x,S) \mid S = \dfrac{108}{x} + \pi x^2 \wedge x \in]0, \infty[\right\}$

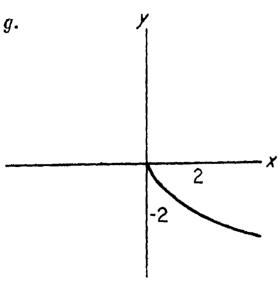
Exercise 18, Page 136

13. $\{(-3,-1)\}$
16. $\{(x,y) \mid 3x - y - 4 = 0 \vee 3x + y = 0\}$
17. $\emptyset$

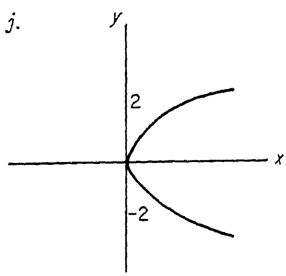
18*f*.



g.



j.



19*f*. $S_6 \cup S_7 = S_{10}$

20*f*. For S_6, $D^* = [0, \infty[$
$R^* = [0, \infty[$

g. For S_7, $D^* = [0, \infty[$
$R^* =]-\infty, 0]$

j. For S_{10}, $D^* = [0, \infty[$
$R^* =]-\infty, \infty[$

22. $\{(\frac{8}{9}, -\frac{4}{3}), (2, 2)\}$

24. $\{(\sqrt{5}, \sqrt{2}), (\sqrt{5}, -\sqrt{2}), (-\sqrt{5}, \sqrt{2}), (-\sqrt{5}, -\sqrt{2})\}$

26. $\{(\frac{6}{7}, -3, \frac{2}{7})\}$

28. $\{(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})\}$

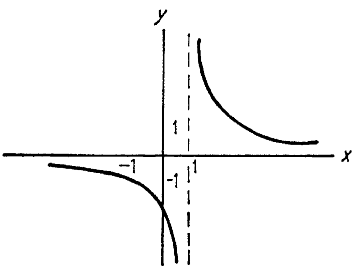
29. $\left\{(-1, -2), \left(-\frac{i}{3}, 0\right)\right\}$

Exercise 19, Page 144

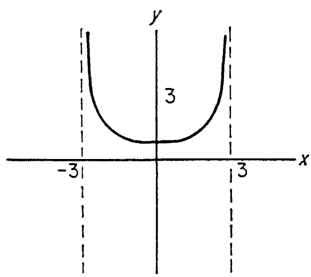
3. $f + g = \{(x,y) \mid y = x^2 + 2x + 1\}$
$f - g = \{(x,y) \mid y = -x^2 + 2x + 1\}$
$fg = \{(x,y) \mid y = 2x^3 + x^2\}$
$\frac{f}{g} = \left\{(x,y) \mid y = \frac{2x + 1}{x^2}\right\}$
$f \circ g = \{(x,y) \mid y = 2x^2 + 1\}$
$f^{-1} = \left\{(x,y) \mid y = \frac{x - 1}{2}\right\}$
$g^{-1} = \{(x,y) \mid y^2 = x\}$

5. $f + g = \{(x,y) \mid y = 3 + \sqrt{2 - x}\}$
$f - g = \{(x,y) \mid y = \sqrt{2 - x} - 3\}$
$fg = \{(x,y) \mid y = 3\sqrt{2 - x}\}$
$\frac{f}{g} = \left\{(x,y) \mid y = \frac{\sqrt{2 - x}}{3}\right\}$
$f \circ g = \{(x,y) \mid y = \sqrt{-1}\} = \emptyset$
$f^{-1} = \{(x,y) \mid x = \sqrt{2 - y}\}$
$g^{-1} = \{(x,y) \mid x = 3\}$

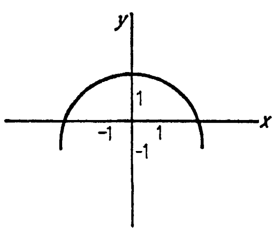
9*a*.



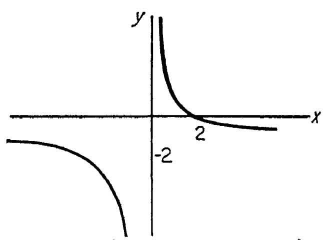
$f \circ g = \left\{(x,y) \mid y = \frac{2}{x - 1}\right\}$



$f \circ h = \left\{(x,y) \mid y = \frac{2}{\sqrt{9 - x^2}}\right\}$



$g \circ h = \{(x,y) \mid y = \sqrt{9 - x^2} - 1\}$



$g \circ f = \left\{(x,y) \mid y = \frac{2}{x} - 1\right\}$

CHAPTER 5

Exercise 20, Page 157

1*a*. Yes
d. No
g. Yes
i. No
k. Yes
m. No

2.

Property	*a*	*b*	*e*	*i*
Closure	Yes	Yes	Yes	Yes
Commutativity	Yes	No	No	No
Associativity	No	No	Yes	No

3*a*. Yes
d. No
f. Yes

Exercise 21, Page 168

2*b*. $A \cap (A \cap B)' = A \cap B'$

A	B	B'	$A \cap B$	$(A \cap B)'$	$A \cap (A \cap B)'$	$A \cap B'$
$\in$	$\in$	$\notin$	$\in$	$\notin$	$\notin$	$\notin$
$\in$	$\notin$	$\in$	$\notin$	$\in$	$\in$	$\in$
$\notin$	$\in$	$\notin$	$\notin$	$\in$	$\notin$	$\notin$
$\notin$	$\notin$	$\in$	$\notin$	$\in$	$\notin$	$\notin$

Columns 6 and 7 are identical.

j. $(A' \cup B)' = (A \cup B') \cap (A \cup B) \cap (A' \cup B')$

A	B	A'	B'	$A \cup B'$	$A \cup B$	$A' \cup B'$	$A' \cup B$	$(A' \cup B)'$	$(A \cup B') \cap (A \cup B) \cap (A' \cup B')$
$\in$	$\in$	$\notin$	$\notin$	$\in$	$\in$	$\notin$	$\in$	$\notin$	$\notin$
$\in$	$\notin$	$\notin$	$\in$	$\in$	$\in$	$\in$	$\notin$	$\in$	$\in$
$\notin$	$\in$	$\in$	$\notin$	$\notin$	$\in$	$\in$	$\in$	$\notin$	$\notin$
$\notin$	$\notin$	$\in$	$\in$	$\in$	$\notin$	$\in$	$\in$	$\notin$	$\notin$

3*a*. (1) $(A \cup B') \cap (A \cup C')$
(3) $(A' \cap C) \cup (B \cap C) \cup (A' \cap D) \cup (B \cap D)$
b. (2) $(A \cup B) \cap (A \cup C)'$
(4) $(A \cup C) \cap (B \cup C) \cap (A \cup D) \cap (B \cup D)$
c. (3) $\emptyset$
(7) A'
(8) $\emptyset$

5*b*. $A \cup (A \cup B)' = A \cup B'$
d. $(A \cup B') \cup B' = A \cup B'$
g. $[A' \cap (B \cap C)]' = A \cup B' \cup C'$

6*a*. $A \cap B' = \{x \in U \mid x \in A \wedge x \notin B\}$
b. $A' \cup B' = \{x \in U \mid x \notin A \vee x \notin B\}$

Exercise 22, Page 189

2a.

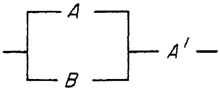
A	B	A'	B'	$A'B'$	$A+B$	$(A'B')'$
1	1	0	0	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	0	0

f.

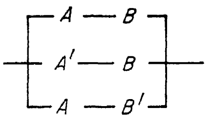
B	C	D	$C+D$	$(C+D)'$	$(C+D)'+B$	$B[(C+D)'+B]$
1	1	1	1	0	1	1
1	1	0	1	0	1	1
1	0	1	1	0	1	1
1	0	0	0	1	1	1
0	1	1	1	0	0	0
0	1	0	1	0	0	0
0	0	1	1	0	0	0
0	0	0	0	1	1	0

4a. Yes
b. No
c. Yes

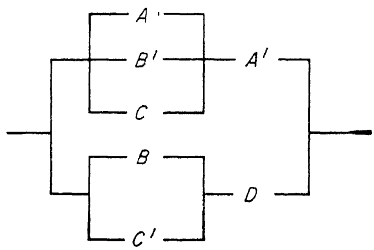
7a.



e.



h.



8b. $A + BC$
d. $AB'(C + B)$
e. $(AB' + C + DE)B$

12a. True
c. True
d. False
f. False
h. True

16b. No
d. No
f. Yes
h. Yes
k. Yes

Exercise 23, Page 200

1a. No
c. No
e. Yes
g. Yes

2a. No
b. No
e. No
i. No

5. Yes; yes

Exercise 24, Page 208

1a. Yes
c. Yes
e. No
g. No
i. Yes

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